UDC 514.18
DOI: 10.15587/1729-4061.2025.343387

ANALYTICAL POINT-FORM DESCRIPTION OF THE TECHNIQUE FOR GRAPHICAL DIFFERENTIATION OF A PLANE CURVE

Viktor Vereschaga
Doctor of Technical Sciences, Professor*
Kseniia Lysenko
Doctor of Philosophy (PhD)*
Yevhen Adoniev
Doctor of Technical Sciences, Associate Professor*
Ernest Murtaziiev
Doctor of Philosophy (PhD), Associate Professor*
Ivan Vereshchaha
Senior Architect
GlobalLogic EMEA
Sheptytskykh str., 26, Lviv, Ukraine, 79016
Tetiana Volina
Corresponding author
Doctor of Technical Sciences, Associate Professor
Department of Descriptive Geometry, Computer Graphics and Design
National University of Life and Environmental Sciences of Ukraine
Heroyiv Oborony str., 15, Kyiv, Ukraine, 03041
E-mail: volina@nubip.edu.ua
*Department of Mathematics and Physics
Bogdan Khmelnitsky Melitopol State Pedagogical University
Naukovoho mistechka str., 59, Zaporizhzhia, Ukraine, 69097

This study considers graphic differentiation, in particular, a chord method, as one of the options for graphic differentiation in terms of replacing graphic operations with analytical ones in point form.

Determining the reference point and the center of projection for constructing a strip of differential projection correlates its positions with respect to the values of the derivative of the function, which is graphically represented by a discrete series of points. The reference point, the right differential projection of the first and left differential projection of the second points have the same values in the field of derivatives. However, they do not coincide with the values of the derivatives of the original functions. To establish such a correspondence, the difference between the left and right differential projections of the first point is divided in half and subtracted from the first derivative of the original function – the point polynomial.

Relative to the reference point, parallel to the first link of the accompanying broken line of the discretely represented curve, a straight line is drawn that intersects the abscissa axis at the center of the projection. Finding the reference point and the projection center is carried out analytically in point form without any graphic operations. Rays are drawn from the projection center parallel to one of the links of the accompanying polyline, thus forming a strip of differential projections, within which the values of the angles of inclination of the tangents to the curve at the base points are selected. Discrete derivative values are connected by straight line segments or remain separate points. The resulting derivative values coincide with the analytical values with a deviation of no more than 0.5–1.5 units.

The developed algorithms could be integrated into automated design and engineering analysis systems for effective calculation of derivatives of discretely given curves. In addition, they could serve as the basis for designing computationally productive modules in artificial intelligence and digital data processing systems that work with geometric and discrete information arrays

Keywords: point polynomial, strip of diffprojections, approximation, analytical chord method, drawings analytization

Received 25.08.2025
Received in revised form 28.10.2025
Accepted 06.11.2025
Published 30.12.2025

How to Cite: *Vereschaga, V., Lysenko, K., Adoniev, Y., Murtaziiev, E., Vereshchaha, I., Volina, T. (2025). Analytical point-form description of the technique for graphical differentiation of a plane curve. Eastern-European Journal of Enterprise Technologies, 6 (1 (138)), 54–63. https://doi.org/10.15587/1729-4061.2025.343387*

1. Introduction

The graphical differentiation method has historically proven its effectiveness [1] for rapid, albeit approximate, engineering analysis and process optimization at the stages of initial design [2, 3].

However, in modern practice, where geometric, kinematic, and functional data are generated and processed exclusively in discrete point form (CAD/CAE, measuring systems), direct application of the graphical differentiation method has become impossible. This is explained by the need to perform resource-intensive graphical constructions, which inevitably introduce significant subjective errors.

The restoration of the practical value of the graphical differentiation method is possible only through its complete analytical formalization and digitization, which will make it possible to replace graphical operations with accurate computational algorithms in point form. There is a particular need for such algorithms in the field of mechanical engineering and kinematics, where it is necessary to quickly calculate the velocities and accelerations (first and second derivatives) of objects whose trajectories are given by discrete points or measurement data. Conventional methods of mathematical analysis often require preliminary approximation, which is inefficient for large arrays of discrete data.

In addition, the integration of computational algorithms for finding derivatives can be carried out in artificial intelligence models and neural networks that work with discrete data. For example, in problems of pattern recognition, geometric data processing, or optimization processes that require iterative refinement of coefficients. Current scientific and practical research and design using artificial intelligence [4, 5] methods evolve reasonably well given the accuracy of the digitized graphic differentiation method.

Among the entire variety of graphical differentiation methods, the most interesting is the chord method. Devising a digitization technique is relevant from the point of view of using in any system of digital transmission of data encoded in discrete signal pulses.

Thus, the relevance of research in this area is predetermined by two key aspects:

– theoretical restoration: analytical representation of the chord method, which replaces resource-intensive graphical operations with high-precision computational ones, thereby restoring the practical value of the graphical differentiation method;

– technological integration: adaptation of this digitized method to modern technological requirements, which allows its effective integration into CAD/CAE systems and artificial intelligence algorithms for rapid analysis of discretely specified functions.

2. Literature review and problem statement

In [6], the results of research on latent variable derivative models of Gaussian processes that can process multidimensional input data using modified covariance derivative functions are reported. The modifications take into account the complexity of the basic data generation process, such as scaled derivatives, variable information in several input dimensions, and interaction between outputs. It is shown that the accuracy of the latent variable estimation can be significantly improved by including derivative information due to the proposed modifications of the covariance function. The issues of practical scaling of the method remain unresolved.

In [7], an approach to shape optimization using the finite element method is given. The geometry is described by a discrete function of a set of levels, and the objective functionals are defined over volume domains. However, the same issues of practical scaling of the method and reducing computational costs for complex geometries remain unresolved since it is the computational complexity that limits its application. These difficulties can be overcome by improving numerical optimization algorithms and using more efficient approximation strategies for functions of a set of levels.

This is the approach used in [8], which reports a polynomial scheme for finding exact solutions to nonlinear partial differential equations based on series expansions and resummation associated with the renormalization group. This approach makes it possible to reduce the problem to solving only linear algebraic equations, avoiding complex analysis methods, and provides the construction of one- and two-soliton, as well as periodic, solutions. At the same time, the issues of scalability of the method and its application to more complex classes of equations remain unresolved, which is limited by the increase in computational complexity. These difficulties can be overcome by improving resummation algorithms and devising more efficient equation reduction procedures.

In [9], methods of geometric modeling and optimization of multidimensional data within the framework of the Radischev integrated drawing system are considered. The authors analyze the possibilities of using such an approach to increase the accuracy and efficiency of processing complex data. In [10], the strength characteristics of high-strength steel-fiber concrete at elevated temperatures are investigated using mathematical modeling. The work is aimed at predicting the behavior of the material under extreme conditions. The authors of [11] proposed an approach to geometric modeling of multifactorial processes and phenomena based on multidimensional parabolic interpolation. The work shows that the method allows for more accurate consideration of complex dependences between parameters. Thus, papers [9–11] consider the construction and analysis of mathematical models for multidimensional space by constructing interpolation curves in point (general) form in parametric form. However, these studies bypass the issues of constructing derivatives for point interpolation curves, not to mention the graphical differentiation of discretely given curves.

In [12], automated control systems for hydraulic locks for flood prevention are considered. The authors propose optimization of the design and control algorithms based on the integration of numerical modeling methods and analysis of hydrodynamic processes. A feature of the study is the use of adaptive algorithms that make it possible to take into account dynamic changes in environmental conditions. However, despite the effectiveness of the proposed solutions, the question of using graphical differentiation methods for analyzing nonlinear dependences in systems with discrete data remains open, which limits the possibilities of further improving the accuracy of forecasting and optimization.

All this allows us to assert that the issue of devising a computational method for approximately finding the first and second derivatives for interpolation curves in parametric form that interpolate the original discretely given flat line curves is unresolved.

Our review of the literature in the field of numerical and geometric differentiation shows that existing methods do not offer a single effective solution for calculating the derivatives of curves given by discrete point polynomials. Numerical differentiation methods often require significant computational resources or preliminary approximation. In contrast, the chord method, despite its simplicity and clarity, remains unsuitable for digital application because of the need for graphical constructions. Thus, an unsolved task is to devise an analytical description of the chord method in point form, which would completely eliminate graphical operations, provide the necessary accuracy, and become a productive tool for modern systems working with discrete data.

3. The study materials and methods

The aim of our study is to devise a technique for analytical representation of the chord method in graphical differentiation of a flat, discretely given curve line by points. This will make it possible to programmatically implement effective methods of graphical differentiation for use in further research using artificial intelligence.

To achieve the goal, the following tasks were set:

– to propose a technique for determining the projection center for correlating the values of derivatives found by graphical differentiation and analytically;

– to develop an algorithm for forming a strip of differential projections and to calculate a test case for digitized graphical differentiation by the method of chords.

4. The study materials and methods

The object of our study is graphical differentiation, in particular, the method of chords, as one of the options for graphical differentiation in terms of replacing graphical operations with analytical ones in point form. The hypothesis of the study assumes that a digitized technique of forming a strip of differential projections for a broken line can be used to select approximate values of derivatives at the nodes of a discretely given plane curve.

In the study, it is assumed that the plane curved line to be differentiated is given exclusively by a discrete series of points (a point polynomial). This is a simplification compared to a continuous function but corresponds to modern digital methods of data representation.

In addition, instead of working with infinitely small increments (as in classical differentiation), the method works with finite chords (segments of the accompanying broken line), which is a fundamental simplification of the chord method.

It is adopted that the studied segments of the curve are regular. The assumption is that any graphical operation of the chord method (construction of chords, drawing parallel lines, finding the centers of projection) can be absolutely exactly replaced by the corresponding system of analytical equations in point form. It is accepted that to establish a correspondence between the values of the derivatives obtained by the graphical method of chords and the values calculated by the methods of mathematical analysis, it is sufficient to apply a specific correction coefficient. This is an empirical or procedural assumption/simplification aimed at "calibrating" the graphical method to the analytical standard.

The chord method can be used as one of the techniques of graphical differentiation. A fragment of the technique to form a strip of differential projections is briefly given below.

Let the initial three points $A_1(2, 1)$; $A_2(4, 3)$; $A_3(8, 2)$ be given in the coordinate system Oxy (Fig. 1). The simplex CAB, whose vertex $C=0$ coincides with the origin of the coordinate system – point O, and two of its points have coordinates $A(8, 0)$; $B(0, 4)$, can be combined with the coordinate system. In Fig. 1, CAB is drawn with thickened straight line segments.

The parameters of points $A_1(p_1, q_1, r_1)$; $A_2(p_2, q_2, r_2)$; $A_3(p_3, q_3, r_3)$ as the coordinate relation take the following form

$$q_i=\frac{y_i}{y_B},\ r_i=1-p_i-q_i,\ \text{for } i=\overline{1,3}. \tag{1}$$

According to (1), we can calculate parameters for A_i, $i=\overline{1,3}$ in simplex CAB:

$$\begin{aligned}&A_1: p_1=\frac{2}{8}=0.25,\ q_1=\frac{1}{4}=0.25,\\ &r_1=1-0.25-0.25=0.5\Rightarrow A_1(0.25;0.25;0.5);\\ &A_2: p_2=0.5,\ q_2=\frac{3}{4}=0.75,\\ &r_2=1-0.5-0.75=-0.25\Rightarrow A_2(0.5;0.75;-0.25);\\ &A_3: p_3=\frac{8}{8}=1,\ q_3=\frac{2}{4}=0.5,\\ &r_3=1-1-0.5=-0.5\Rightarrow A_3(1;0.5;-0.5).\end{aligned} \tag{2}$$

The point equations of points A_i, $i=\overline{1,3}$ in simplex CAB take the following form:

$$\left.\begin{aligned}A_1&=Ap_1+Bq_1+Cr_1,\\ A_2&=Ap_2+Bq_2+Cr_2,\\ A_3&=Ap_3+Bq_3+Cr_3.\end{aligned}\right] \tag{3}$$

Point equations (3) are a calculation scheme in coordinate form:

$$\left.\begin{aligned}A_1&\Rightarrow\begin{cases}x_{A_1}=x_A\cdot p_1+x_B\cdot q_1+x_C\cdot r_1=8\cdot0.25+0\cdot0.25+0\cdot0.5=2;\\ y_{A_1}=y_A\cdot p_1+y_B\cdot q_1+y_C\cdot r_1=0\cdot0.25+4\cdot0.25+0\cdot0.5=1;\end{cases}\\ A_2&\Rightarrow\begin{cases}x_{A_2}=x_A\cdot p_2+x_B\cdot q_2+x_C\cdot r_2=8\cdot0.5+0\cdot0.75+0\cdot(-0.25)=4;\\ y_{A_2}=y_A\cdot p_2+y_B\cdot q_2+y_C\cdot r_2=0\cdot0.5+4\cdot0.75+0\cdot(-0.25)=3;\end{cases}\\ A_3&\Rightarrow\begin{cases}x_{A_3}=x_A\cdot p_3+x_B\cdot q_3+x_C\cdot r_3=8\cdot1+0\cdot0.5+0\cdot(-0.5)=8;\\ y_{A_3}=y_A\cdot p_3+y_B\cdot q_3+y_C\cdot r_3=0\cdot1+4\cdot0.5+0\cdot(-0.5)=2.\end{cases}\end{aligned}\right] \tag{4}$$

The coordinates of points A_i, $i=\overline{1,3}$ calculated in (4) coincide with the values of the corresponding points – this means that parameters p_i, q_i, r_i, $i=\overline{1,3}$ from (1) were calculated correctly.

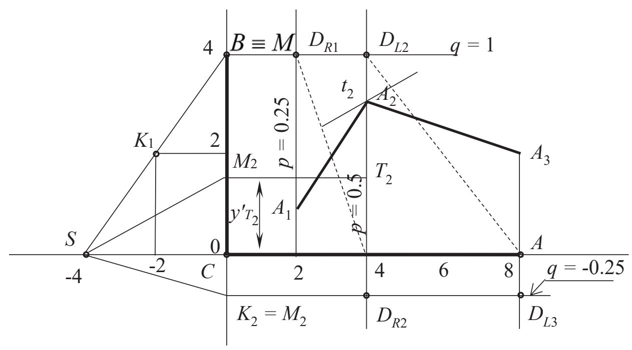

Fig. 1. Scheme for constructing diffprojections D_{L1}, D_{R2}, D_{L2}, D_{R3}

To construct differential projections D_{L1}, D_{R2}, D_{L2}, D_{R3} it is necessary to calculate the projection pole S (Fig. 1). Let the pole have coordinates $S(-4, 0)$.

It is necessary to draw segment $SK_1 \parallel A_1A_2$, using point equation $K_1=S+A_2-A_1$, which in coordinate form is calculated as follows:

$$\left.\begin{aligned}x_{K_1}&=x_S+x_{A_2}-x_{A_1}=-4+4-2=-2\\ y_{K_1}&=y_S+y_{A_2}-y_{A_1}=0+3-1=2\end{aligned}\right\}\Rightarrow$$
$$\Rightarrow K_1(-2;2)\Rightarrow SK_1\parallel A_1A_2. \tag{5}$$

At point M, straight line SK_1 intersects the axis Oy. The parameters of points S and K_1 in simplex CAB are calculated as follows:

$$\begin{aligned}&p_S=\frac{x_S}{x_A}=-0.5;\ q_S=\frac{y_S}{y_B}=0;\\ &p_{K_1}=\frac{x_{K_1}}{x_A}=-0.25;\ q_{K_1}=\frac{y_{K_1}}{y_B}=0.5.\end{aligned} \tag{6}$$

Thus, $S(p_S;q_S), K_1(p_{K_1};q_{K_1})$.

It is necessary to parameterize points S and K_1 through the vertices of simplex CAB:

$$S = Ap_S + Bq_S,$$

$$K_1 = Ap_{K_1} + Bq_{K_1}. \quad (7)$$

The current point M on the straight line SK_1 is determined from the following point equation [1]

$$M = (K_1 - S)u + S, \quad (8)$$

or in expanded form taking into account (7)

$$M = (Ap_{K_1} + Bq_{K_1} - Ap_S - Bq_S)u + Ap_S + Bq_S = \\ = A\left[(p_{K_1} - p_S)u + p_S\right] + B\left[(q_{K_1} - q_S)u + q_S\right]. \quad (9)$$

Point M will belong to the Oy axis if expression

$$(p_{K_1} - p_S)u + p_S = 0 \Rightarrow u = \frac{p_S}{p_S - p_{K_1}},$$

then

$$M = B\left[(q_{K_1} - q_S)\cdot\frac{p_S}{p_S - p_{K_1}} + q_S\right] = \\ = B\cdot\frac{q_{K_1}\cdot p_S - q_S\cdot p_{K_1}}{p_S - p_{K_1}} = B\cdot\frac{0.25}{0.25} = B, \quad (10)$$

as can be seen from Fig. 1, points M and B coincide.

On a horizontal line with parameter $q = 1$ (Fig. 1), passing through point M, for point A_1 a diffprojection can be found: $D_{R_1}(0.25;1)$, for point A_2 – diffprojection $D_{L_2}(0.5;1)$.

Similar to point K_1, it is necessary to find point K_2

$$K_2 = S + A_3 - A_2,$$

or in coordinate form:

$$\left.\begin{aligned} x_{K_2} &= x_S + x_{A_3} - x_{A_2} = -4 + 8 - 4 = 0 \\ y_{K_2} &= y_S + y_{A_3} - y_{A_2} = 0 + 2 - 3 = -1 \end{aligned}\right\} \Rightarrow \\ \Rightarrow K_2(0;-1) \Rightarrow SK_1 \parallel A_2A_3. \quad (11)$$

Since $x_{K_2} = 0$, points $K_2 = M_2$ coincide (Fig. 1).

On the horizontal line with parameter $q = -0.25$ (Fig. 1), passing through point M_2, for point A_2 we can find diffprojection $D_{R_2}(0.5;-0.25)$, for point A_3 – diffprojection $D_{L_3}(1;-0.25)$. By connecting the dashed lines of the corresponding diffprojections, a strip of diffprojections can be obtained for the polyline $A_1A_2A_3$. For point A_2, point $T_2 = (D_{R_2} + D_{L_2})/2$ can be chosen for the value $x_2 = 4$ (as one of the possible options). The y'_{T_2} value corresponds to the angle of inclination of tangent t_2 and ray SM_2 (Fig. 1), therefore, $t_2 \parallel SM_2$.

5. Results of calculating the values of the first derivatives at the points of a discretely given plane curve

5. 1. Determining the reference point and the center of projection for the strip of diffprojections

Let the initial discretely given plane curve (DGC) be given by five points (Table 1).

Table 1

Points of initial DGC

A_i	A_1	A_2	A_3	A_4	A_5
x_i	1	4	1	14	18
y_i	2	8	9	–1	–8

The general form of the point polynomial that interpolates the original DGC is written as follows [2, 3]

$$M(t) = \sum_{i=1}^{n=5} A_i p_i(t), t_1 \le t \le t_5, \quad (12)$$

which in expanded (coordinate) form is written as

$$M_x(t) = \sum_{i=1}^{5} x_i p_i(t), M_y(t) = \sum_{i=1}^{5} y_i p_i(t). \quad (13)$$

In entries (12) and (13), $p_i(t)$ for $i = \overline{1,5}$ are the basis functions of the point polynomial and are the invariants of parallel projection, which are formed as follows

$$p_i(t) = \frac{\prod\limits_{\substack{i=1 \\ i\ne(i)}}^{n=5}(t_i - t)}{\prod\limits_{\substack{i=1 \\ i\ne(i)}}^{n=5}(t_i - t_{(i)})}, t_1 \le t \le t_5. \quad (14)$$

In the original notation (14) the functional basis of the point polynomial (12) is given. The index notation $i = (i)$ under the product sign in both the numerator and denominator means that the difference with parameter t_i cannot be used in the products of differences. Its index coincides with the index, which is taken in brackets to distinguish in the notation of the basis functions $p_{(i)}(t)$.

To determine parameters t_i, $i = \overline{1,5}$, which are part of the basis functions (14) of the point polynomial (12), the lengths of each of the links of the accompanying broken line (ABL), which is constructed at the starting points of DGC (Table 1), must be calculated. In general, the lengths of these links are denoted $l_{i,i-1}$, $i = \overline{2,5}$. It should be noted that $l_{1,0} = 0$. Therefore, for a plane curve

$$l_{i,i-1} = \sqrt{(x_{A_i} - x_{A_{i-1}})^2 + (y_{A_i} - y_{A_{i-1}})^2}, i = \overline{2,5}. \quad (15)$$

The results of calculations of the lengths of the accompanying broken line (ABL) links are given in Table 2.

Let the sum of the first three segments be unity relative to measurement L_e

$$L_e = l_{1,0} + l_{2,1} + l_{3,2} = 12.7909664627. \quad (16)$$

The values of parameters t_i for all initial points A_i; $i = \overline{1,5}$ can be calculated taking into account that $t_1 = 0$

$$t_{(i)} = \frac{\sum\limits_{i=2}^{\tau} l_{i,i-1}}{L_e}, i = \overline{2,5}, \tau = \overline{2,(i)}; i = (i). \quad (17)$$

It is necessary to first calculate the numerator (Table 3).

The results of the calculations of parameters t_i are given in Table 4.

Table 2

ABL links length

i	1	2	3	4	5
$l_{i,\,i-1}$	0	6.70820393249	6.08276253029	10.7703296142	8.06225774829

Table 3

Cumulative length of ABL

$I,\, i-1$	1	2	3	4	5
$\sum\limits_{i=2}^{5}$	0	6.70820393249	6.08276253029	10.7703296142	8.06225774829

Table 4

Values of parameters t_i

A_i	A_1	A_2	A_3	A_4	A_5
t_i	0	0.52444855922	1.0	1.84202625701	2.47233498088

By substituting the values of parameters t_i from Table 4 into the basis function (14) and then into the polynomial (12), we can obtain the equation of the point polynomial that interpolates the original five points (Table 1)

$$\begin{aligned} M_y(t) = {} & y_1\frac{t^4 - t^3\cdot 5.83880979711 + t^2\cdot 11.6555762831 - t\cdot 9.20516079054 + 2.38839430447}{2.38839430447} - \\ & - y_2\frac{t^3 - t^3\cdot 5.31436123789 + t^2\cdot 8.86846718879 - t\cdot 4.5541059509}{0.64008885246} + \\ & + y_3\frac{t^4 - t^3\cdot 4.83880979711 + t^2\cdot 6.81676648606 - t\cdot 2.38839430447}{0.58956238448} - \\ & - y_4\frac{t^4 - t^3\cdot 3.9967835401 + t^2\cdot 4.29339605873 - t\cdot 1.29661251863}{1.28810422298} + \\ & + y_5\frac{t^4 - t^3\cdot 3.36647481623 + t^2\cdot 3.33252283276 - t\cdot 0.96604801653}{4.46921134473}. \end{aligned} \tag{18}$$

The equation of the first derivative with respect to parameter t for polynomial (18) will take the following form

$$\begin{aligned} M'_y(t) = {} & y_1\frac{4t^3 - 3t^2\cdot 5.83880979711 + 2t\cdot 11.6555762831 - 9.20516079054}{2.38839430447} - \\ & - y_2\frac{4t^3 - 3t^2\cdot 5.31436123789 + 2t\cdot 8.86846718879 - 4.5541059509}{0.64008885246} + \\ & + y_3\frac{4t^3 - 3t^2\cdot 4.83880979711 + 2t\cdot 6.81676648606 - 2.38839430447}{0.58956238448} - \\ & - y_4\frac{4t^3 - 3t^2\cdot 3.9967835401 + 2t\cdot 4.29339605873 - 1.29661251863}{1.28810422298} + \\ & + y_5\frac{4t^3 - 3t^2\cdot 3.36647481623 + 2t\cdot 3.33252283276 - 0.96604801653}{4.46921134473}. \end{aligned} \tag{19}$$

The value of the first derivative (19) $M'_y(t)$ at point A_1, for which $t_1 = 0$, will take the following form

$$\begin{aligned} M'_y(t) = {} & -2\cdot\frac{9.20516079054}{2.38839430447} + 8\cdot\frac{4.5541059509}{0.64008885246} - \\ & - 9\cdot\frac{2.38839430447}{0.58956238448} - 1\cdot\frac{1.29661251863}{1.28810422298} + \\ & + 8\cdot\frac{0.96604801653}{4.46921134473} = 13.4726482308 \approx 13.47. \end{aligned} \tag{20}$$

The vertices of simplex CAB (Fig. 2) have the following coordinates: $C(0;0)$; $A(10;0)$; $B(0;2)$. An imaginary center of projection $\bar{S}(-4;0)$, can be chosen arbitrarily on the Ox axis, i. e., always to the left of the Oy axis (Fig. 2, upper part).

For the segment (A_1A_2), coordinates $\bar{x}_{k_1}$ and $\bar{y}_{k_1}$ of approximate point $\bar{k}_1$ are determined as follows:

$$\left.\begin{aligned} & \bar{x}_{k_1} = x_{\bar{S}} + x_2 - x_1 = \\ & = -4 + 4 - 1 = -1, \\ & \bar{y}_{k_1} = y_{\bar{S}} + y_2 - y_1 = \\ & = 0 + 8 - 2 = 6 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \bar{k}_1\left(\bar{x}_{k_1};\bar{y}_{k_1}\right) = = \bar{k}_1(-1;6). \tag{21}$$

Similarly for the section (A_2A_3):

$$\left.\begin{aligned} & \bar{x}_{k_2} = x_{\bar{S}} + x_3 - x_2 = \\ & = -4 + 10 - 4 = 2, \\ & \bar{y}_{k_2} = y_{\bar{S}} + y_3 - y_2 = \\ & 0 + 9 - 8 = 1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \bar{k}_2\left(\bar{x}_{k_2};\bar{y}_{k_2}\right) = = \bar{k}_2(2;1). \tag{22}$$

For $\bar{k}_1$ and $\bar{k}_2$ in the CAB simplex, it is necessary to calculate parameters $\bar{p}_{k_1}$, $\bar{q}_{k_1}$ and $\bar{p}_{k_2}$, $\bar{q}_{k_2}$, accordingly:

$$\bar{p}_{k_1} = \frac{\bar{x}_{k_1}}{x_A} = \frac{-1}{10} = -0.1;$$

$$\bar{q}_{k_1} = \frac{\bar{y}_{k_1}}{y_B} = \frac{6}{2} = 3;$$

$$\bar{p}_{k_2} = \frac{\bar{x}_{k_2}}{x_A} = \frac{2}{10} = 0.2;$$

$$\bar{q}_{k_2} = \frac{\bar{y}_{k_2}}{y_B} = \frac{1}{2} = 0.5. \tag{23}$$

Using (10), points $\bar{M}_1$ and $\bar{M}_2$ can be found on the Oy axis:

$$\left.\begin{aligned} & \bar{y}_{M_1} = y_B\cdot\frac{\bar{p}_S\cdot\bar{q}_{k_1}}{\bar{p}_S - \bar{p}_{k_1}} = 2\cdot\frac{(-0.4)\cdot 3}{(-0.4)-(-0.1)} = 8, \\ & \bar{y}_{M_2} = y_B\cdot\frac{\bar{p}_S\cdot\bar{q}_{k_2}}{\bar{p}_S - \bar{p}_{k_2}} = 2\cdot\frac{(-0.4)\cdot 0.5}{(-0.4)-0.2} = 0.(6) \approx 0.67 \end{aligned}\right]. \tag{24}$$

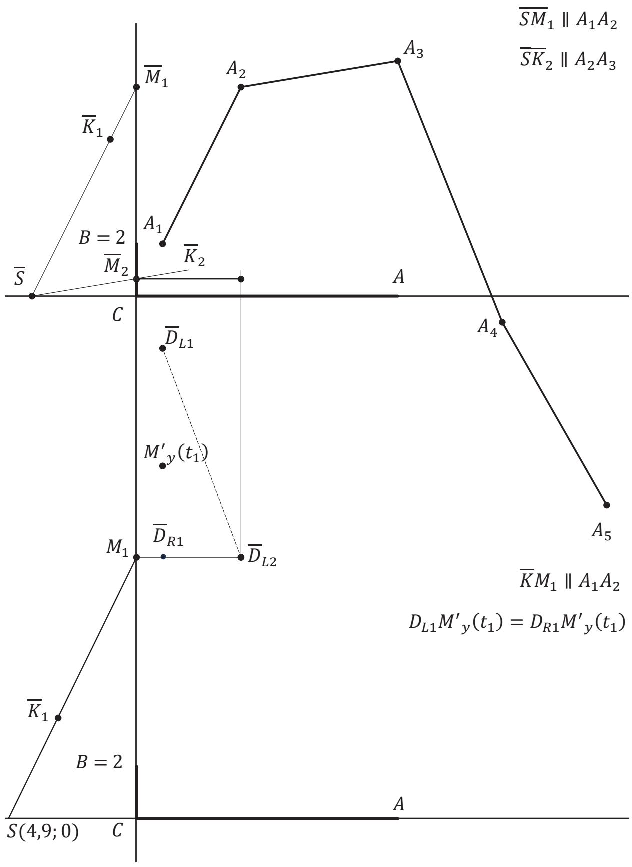

Fig. 2. Construction of reference point M_1 and projection center S

The absolute value $\Delta = \left|\bar{y}_{M_1} - \bar{y}_{M_2}\right|$ is:

$$\Delta = \left|8 - 0.67\right| = 7.33;$$

$$\frac{\Delta}{2} \approx 3.665. \qquad (25)$$

Reducing the value of the first traditional derivative $M'_y(t_1)$ from (20) by $\Delta/2$ gives the result

$$y_{R_1} = y_{L_2} = M'_y(t_1) - \frac{\Delta}{2} = 13.473 - 3.665 = 9.808 \approx 9.81. \qquad (26)$$

By increasing value $M'_y(t_1)$ by $\Delta/2$, we can obtain y_{L_1}

$$y_{L_1} = M'_y(t_1) + \frac{\Delta}{2} = 13.473 + 3.665 = 17.138 \approx 17.14. \qquad (27)$$

Hence, coordinates of three differential projections can be obtained:

$$D_{L_1}(x_1; y_{L_1}) \Rightarrow D_{L_1}(1;17.14);$$

$$D_{R_1}(x_1; y_{R_1}) \Rightarrow D_{R_1}(1;9.81);$$

$$D_{L_2}(x_2; y_{L_2}) \Rightarrow D_{L_2}(4;9.81). \qquad (28)$$

From Fig. 2: $y_{R_1} = y_{L_2} = y_{M_1}$. So, the reference point M_1 will have coordinates $x_{M_1} = 0, y_{M_1} = 9.81$, i.e., $M_1(0;9.81)$.

It is necessary to calculate point K_1, which is the fourth vertex of parallelogram $A_1A_2M_1K_1$, for which $M_1K_1 \parallel A_1A_2$. Point form for calculation will take the following form: $K_1 = M_1 + A_1 - A_2$. It is expanded into coordinate notation as follows:

$$x_{K_1} = x_{A_1} + x_{M_1} - x_{A_2} = 1 + 0 - 4 = -3,$$

$$y_{K_1} = y_{A_1} + y_{M_1} - y_{A_2} = 2 + 9.81 - 8 = -3.81. \qquad (29)$$

Thus, the auxiliary point $K_1(-3;3.81)$ was obtained. To form differential projections for all initial points, it is necessary to calculate projection center S on the Ox axis. In point form, $S = M_1K_1 \cap Ox$.

The point equation of straight line M_1K_1, on which point S is located, takes the form

$$S = (M_1 - K_1)u + K_1, \qquad (30)$$

where u is the current parameter along line M_1K_1.

Points M_1 and K_1 in simplex CAB are defined by:

$$M_1 = Ap_{M_1} + Bq_{M_1},$$

$$K_1 = Ap_{K_1} + Bq_{K_1}, \qquad (31)$$

where:

$$p_{M_1} = \frac{x_{M_1}}{x_A} = \frac{0}{10} = 0; \; q_{M_1} = \frac{y_{M_1}}{y_B} = \frac{9.81}{2} = 4.91;$$

$$p_{K_1} = \frac{x_{K_1}}{x_A} = \frac{-3}{10} = -0.3; \; q_{K_1} = \frac{y_{K_1}}{y_B} = \frac{3.81}{2} = 1.91.$$

Substitution (31) in (30) gives the result

$$S = \left(Ap_{M_1} + Bq_{M_1} - Ap_{K_1} - Bq_{K_1}\right)U + Ap_{K_1} + Bq_{K_1} =$$
$$= A\left[\left(p_{M_1} - p_{K_1}\right)U + p_{K_1}\right] + B\left[\left(p_{M_1} - p_{K_1}\right)U + q_{K_1}\right]. \qquad (32)$$

Taking into account that in (32) $y_S = 0$

$$\left(q_{M_1} - q_{K_1}\right)u + q_{K_1}, \text{ hence } u = \frac{q_{K_1}}{q_{K_1} - q_{M_1}}. \qquad (33)$$

Substitution (33) in (32) allows one to write

$$S = A\left[\left(p_{M_1} - p_{K_1}\right)\frac{q_{K_1}}{q_{K_1} - q_{M_1}} + p_{K_1}\right] + B \cdot 0. \qquad (34)$$

Taking into account that $p_{M_1} = 0$ from (31), the end result takes the form

$$S = A\left[\frac{-p_{K_1} \cdot q_{K_1}}{q_{K_1} - q_{M_1}} + p_{K_1}\right], \qquad (35)$$

or in coordinate form

$$x_S = x_A \cdot \left(\frac{-p_{K_1} \cdot q_{K_1}}{q_{K_1} - q_{M_1}} + p_{K_1}\right) =$$
$$= 10 \cdot \left(\frac{0.3 \cdot 1.91}{1.91 - 4.91} + 0.3\right) = -4.91. \qquad (36)$$

Thus, $S(-4.91;0)$.

Determining the reference point M_1 and the projection center S are mandatory for the subsequent construction of the strip of differential projections of the accompanying polyline.

Finding these points brings the values of the derivatives found by graphical differentiation using the chord method into line with the values of traditional derivatives calculated by mathematical analysis methods.

5. 2. Algorithm for constructing a strip of diffprojections, a test case for digitizing graphic differentiation by the chord method

A strip of diffprojections is built on the basis of an accompanying polyline, the vertices of which are discretely given points of a regular plane curved line. It is a part of a plane bounded by straight lines, the inclination angles of which correspond to the inclination angles of each of the links of ABL. The selection of tangents to the desired curve within the strip of diffprojections will not cause the probable appearance of unnecessary inflection points on this desired curved line.

Let a flat discretely given curved line be given by twelve points A_i; $i=\overline{1,12}$, the coordinates of which are given in Table 5.

Table 5

Coordinates of the base points of original DGC

A_i	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
x_i	1	2	4	6	10	12	14	16	18	21	24	25
y_i	2	5	8	10	9	5	–1	–5	–8	–9	–5	–1

The algorithm is as follows:

1. For parameterization, simplex CAB must be chosen, which coincides with the original coordinate system Oxy; therefore: $C(0;0)$; $A(10;0)$; $B(0;2)$.

2. Parameters p_i, q_i for points A_i; $i=\overline{1,12}$ are

$$p_i=\frac{x_i}{x_A},\ q_i=\frac{y_i}{y_A},\ \text{for } i=\overline{1,12}. \quad (37)$$

The results of the calculations are given in Table 6.

Table 6

Parameterization of points A_i in the CAB simplex

A_i	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
p_i	0.1	0.2	0.4	0.6	1	1.2	1.4	1.6	1.8	2.1	2.4	2.5
q_i	1	2.5	4	5	5.5	2.5	–0.5	–2.5	–4	–4.5	–2.5	–0.5

3. It is necessary to calculate the reference point M_i, $i=\overline{1,11}$ and the projection center S. To simplify the calculations, $S(-4;0)$ was taken or relative to simplex CAB in parameters: $p_S=x_S/x_A=-4/10=-0.4$; $q_S=\overline{0/x_B}=0$: $S(-0.4;0)$.

4. For each of the links A_iA_{i+1} for $i=\overline{1,11}$ by constructing parallelograms $A_iA_{i+1}SK_i$ by calculating them from the point equations, it is necessary to find x_{K_1} and y_{K_1}

$$K_i=S+A_{i+1}-A_i\Rightarrow\begin{cases}x_{K_i}=x_S+x_{i+1}-x_i\\ y_{K_i}=y_S+y_{i+1}-y_i\end{cases}, i=\overline{1,11}, \quad (38)$$

and similarly in parametric form

$$K_i=S+A_{i+1}-A_i\Rightarrow\begin{cases}p_{K_i}=p_S+p_{i+1}-p_i\\ q_{K_i}=q_S+q_{i+1}-q_i\end{cases}, i=\overline{1,11}. \quad (39)$$

5. Similarly to (24), coordinates y_{M_i} for points M_i on the Oy axis can be found

$$y_{M_1}=y_B\cdot\frac{p_S\cdot q_{K_1}}{p_S-p_{K_1}}=2\cdot\frac{(-0.4)\cdot q_{K_1}}{(-0.4)-p_{K_1}}=$$
$$=\frac{(-0.8)\cdot q_{K_1}}{(-0.4)-p_{K_1}},\ i=\overline{1,11}\left(i=\overline{1,n}\right). \quad (40)$$

The results of the y_{M_i} coordinate calculations are given in Table 7; and for point M_i the construction is carried out in Fig. 3. The results of coordinates and parameters calculations (17), (18) for all links A_iA_{i+1} of ABL are given in Table 7.

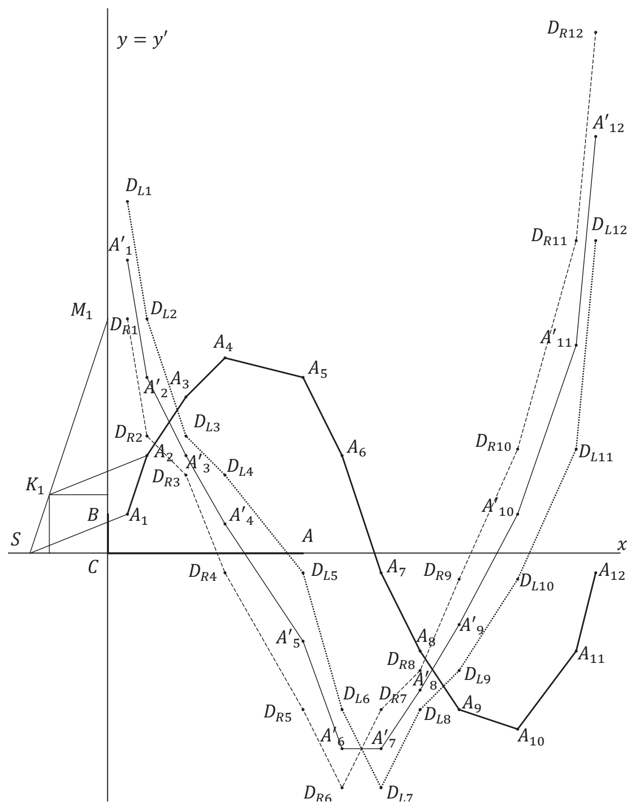

Fig. 3. Construction of a strip of differential projections and determination of composite derivatives at the base points of a discretely given plane curve

Table 7

Calculation of coordinates and parameters of points K_i and M_i

A_iA_{i+1}	A_1A_2	A_2A_3	A_3A_4	A_4A_5	A_5A_6	A_6A_7	A_7A_8	A_8A_9	A_9A_{10}	$A_{10}A_{11}$	$A_{11}A_{12}$
x_{K_i}	–3	–2	–2	0	–2	–2	–2	–2	–1	–1	–3
y_{K_i}	3	3	2	–1	–4	–6	–4	–3	–1	4	4
p_{K_i}	–0.3	–0.2	–0.2	0	–0.2	–0.2	–0.2	–0.2	–0.1	–0.1	–0.3
q_{K_i}	1.5	1.5	1	–0.5	–2	–3	–2	–1.5	–0.5	2	2
y_{M_i}	12	6	4	–1	–8	–12	–8	–6	–1.(3)	5.(3)	16

6. The reference point M_i and the differential projections D_{R_i}, D_{L_i+1} have the same coordinates: $y_{M_i} = y_{D_{Ri}} = y_{D_{Li+1}}$. Therefore, $D_{Ri}\left(x_{A_i}; y_{M_i}\right)$, $D_{Li+1}\left(x_{A_{i+1}}; y_{M_i}\right)$. It is necessary to compile Tables 8, 9 of the differential projection calculations.

In Table 8, the D_{R12} value is missing; in Table 9, the D_{L1} value is missing. They can be found as the sides of a parallelogram.

7. The values of diffprojection D_{Li} are calculated by drawing a segment $D_{L2}D_{L1} \parallel D_{R1}D_{R2}$ (Fig. 3)

$$D_{L1} = D_{R1} + D_{L2} - D_{R2} \Rightarrow y_{L1} = y_{R1} + y_{L2} - y_{R2} \Rightarrow$$
$$\Rightarrow y_{L1} = 12 + 12 - 6 = 18. \qquad (41)$$

8. The values of diffprojection D_{R12} can be found by drawing a segment $D_{L11}D_{L12} \parallel D_{R11}D_{R12}$ (Fig. 3)

$$D_{R12} = D_{R11} + D_{L12} - D_{L11} \Rightarrow y_{R12} = y_{R11} + y_{L12} - y_{L11} \Rightarrow$$
$$\Rightarrow y_{R12} = 16 + 16 - 5.(3) = 26.(6). \qquad (42)$$

9. The arithmetic mean of the first derivatives $A_i', i = \overline{1,12}$ for each of the base points $A_i, i = \overline{1,12}$ is calculated as follows

$$A_i = \frac{D_{Ri} + D_{Li}}{2} \Rightarrow y_i' = \frac{y_{Ri} + y_{Li}}{2}, i = \overline{1,12}. \qquad (43)$$

The results of the calculations corresponding to (43) are given in Table 10.

of derivatives at the nodes of the desired continuous interpolation curve line.

Unlike [9–11], the main feature of the proposed technique is that any initial geometric object is previously given by a composition of base points. They are selected on this object and become the basis for solving the problem. On the plane, simplex *CAB* is arbitrarily selected, relative to which the base points are parameterized in point form, that is, in the general form (3) with respect to their coordinates. On the plane, point forms (3) are revealed in two coordinate records. In spaces of higher dimensions, the number of coordinate entries will correspond to the dimensions of these spaces. In expressions (3) and (4), parameters (1), (14) remain unchanged because they are invariants of parallel projection.

A segment of any regular curve line can be represented discretely by base points (Fig. 1). Through them, an accompanying polyline ($A_1A_2A_3$) can be drawn. Based on it, using the graphical method of chords through the application of differential projections, one can find approximate values of derivatives at the base points of this curve (8) to (10).

Owing to the use of point forms (12), (13), graphical differentiation methods are easily amenable to the development of computational algorithms (digitization). This provides significant advantages in finding values compared to the use of conventional differentiation methods from mathematical analysis (18), (19).

Table 8

Coordinates of right differential projections D_{Ri}

D_{Ri}	D_{R1}	D_{R2}	D_{R3}	D_{R4}	D_{R5}	D_{R6}	D_{R7}	D_{R8}	D_{R9}	D_{R10}	D_{R11}	D_{R12}
x_{Ri}	1	2	4	6	10	12	14	16	18	21	24	25
y_{Ri}	12	6	4	–1	–8	–12	–8	–6	–1.(3)	5.(3)	16	–

Table 9

Coordinates of right differential projections D_{Li}

D_{Li}	D_{L1}	D_{L2}	D_{L3}	D_{L4}	D_{L5}	D_{L6}	D_{L7}	D_{L8}	D_{L9}	D_{L10}	D_{L11}	D_{L12}
x_{Li}	1	2	4	6	10	12	14	16	18	21	24	25
y_{Li}		12	6	4	–1	–8	–12	–8	–6	–1.(3)	5.(3)	16

Table 10

Calculating y_i' derivatives for the points of DGC obtained using the strip of diffprojections

A_i	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
x_i	1	2	4	6	10	12	14	16	18	21	24	25
y_{Ri}	12	6	4	–1	–8	–12	–8	–6	–1.(3)	5.(3)	16	26.(6)
y_{Li}	18	12	6	4	–1	–8	–12	–8	–6	–1.(3)	5.(3)	16
y_i'	15	9	5	1.5	–4.5	–10	–10	–7	–3.65	2	10.65	21.33

Problems using artificial intelligence have solutions in the form of a set of coefficients, the gradual refinement of which through training brings the solution to the problem closer (object recognition). The graphical technique of approximate finding of derivatives meets these requirements and is easily digitized at insignificant resource consumption. This provides significant advantages in solving problems using derivatives.

The determination of reference point $M_1(0;9.81)$ from (26) and the projection center $S(-4.91;0)$ from (35), (36) in Fig. 2 correlate the traditional values of the first derivative (19), (20) with the results from the digitized technique of graphical dif ferentiation. Without correlation between the values of the derivatives obtained by the conventional method and digitized by the chord method, the nature of change in the form of the digitized and graphical traditional derivatives will be the same, but their values will differ.

The proposed technique for analytical representation of graphical differentiation in point form is quite effective in solving individual problems using artificial intelligence.

6. Results of investigating the proposed technique for analytical representation of graphic differentiation: discussion

The proposed method for digitizing graphic differentiation by the chord method has been used to confirm the hypothesis of our study. A strip of diffprojections was constructed and applied to form an algorithm for selecting approximate values

In our test example (Table 5), the initial composition of points is given, the graphic image of which is shown by the main line (Fig. 3), and which is parameterized (Table 7) with respect to simplex *CAB*. The right (Table 8) and left (Table 9) diffprojections are calculated, which are depicted in Fig. 3 (dashed lines). The results of calculating the derivatives (43), which are obtained by the digitized graphical method of chords, are given in Table 10.

The proposed technique of graphical differentiation analytically formalized in point form cannot be applied to segments of curved lines containing inflection points of the 1st and 2nd kind, breaking points, curve discontinuity points, and asymptotic points. However, if at these points the curve segment is divided into two separate ones, on which all points will be regular, then it will be possible to apply the proposed analyzed graphical differentiation to each of these segments. Curves are given in parametric form. Therefore, when points of inflection, self-tangency, or nodal points occur on the segments of curved lines, the method of graphical differentiation analytically formalized in point forms can be applied to them. In this case, only one requirement is put forward: each of the specified special points must be given by two separate points that have converged into one.

The proposed analytically formalized method of graphical differentiation bridges the gap in applying different approaches and techniques to the differentiation of functions, which imposes certain restrictions on their use in modern technologies and increases resource consumption. At the same time, the analytically formalized method of graphical differentiation is the only algorithm that can be applied in the plane to the graphs of any functions of one variable, taking into account the specified restrictions.

The need to calculate the basis functions (14), the point polynomial (12), and its derivative (19) for the parameter value $t=0$ is a drawback of our study.

Further advancement of the analytically formalized graphical differentiation may involve its extension to spatial discretely given segments of regular curves. They generalize processes and are more widespread in modern technologies.

7. Conclusions

1. The parameter records $p_i(t)$ for $i=\overline{1,n}$ devised in a general form build functions for the basis of the interpolation point polynomial. The proposed methodology of parameterization of the accompanying broken line allows us to determine reference point M_1, the y'_{M_1} coordinate of which coincides with coordinates $y'_{D_{R1}}$ and $y'_{D_{L2}}$ of the corresponding diffprojections. Point M_1 makes it possible to find a projection center for calculating diffprojections at all points of the accompanying broken line, which are correlated with the values of the derivatives at its vertices belonging to the segment of a flat continuous regular curve. The peculiarity of our result is that it could be applied to the discrete point representation of graphs of any single-valued or multi-valued functions.

2. An algorithm for the analytically formalized representation of graphical differentiation in point form has been proposed, which is based on the devised parameterization methodology. Within the algorithm, the reference point and the projection center are determined, the coordinates and parameters for all points K_i and M_i, $(i=\overline{1,12})$, are calculated, coordinates for the right D_{Ri} and left D_{Li} $(i=\overline{1,12})$, diffprojections of the accompanying polyline are determined. Based on the values of D_{Ri} and D_{Li} $(i=\overline{1,12})$, the values of the derivatives at all $(i=\overline{1,12})$ base points are selected within the strip of diffprojections. A feature of the developed algorithm of analytically formalized graphic differentiation is that it does not require graphic operations to be performed for its implementation. The figures given in the text of our paper are for informational purposes only.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

Funding

The study was conducted without financial support.

Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Acknowledgments

The authors express their gratitude to the defenders of Ukraine for the opportunity to live and engage in scientific activities.

References

1. Fischer, M., Krause, C. M. (2025). Pivotal examples in graphical differentiation – an analysis of semiotic and theoretic control. Proceedings of the 48th Conference of the International Group for the Psychology of Mathematics Education: Research Reports, 1, 259–266. Available at: https://www.researchgate.net/publication/392626925_PIVOTAL_EXAMPLES_IN_GRAPHICAL_DIFFERENTIATION_-AN_ANALYSIS_OF_SEMIOTIC_AND_THEORETIC_CONTROL
2. Zakharova, I., Shchetynin, S., Shchetynina, V., Zusin, A., Volenko, I. (2025). Use of robotic and automated systems in welding and restoration of parts. Machinery & Energetics, 16 (1), 117–129. https://doi.org/10.31548/machinery/1.2025.117
3. Aghayeva, K., Krauklit, G. (2025). Automated methane emission monitoring systems based on satellite data: Radiation transfer model analysis. Machinery & Energetics, 16 (1), 146–156. https://doi.org/10.31548/machinery/1.2025.146
4. Turchyn, O. (2024). Introduction of neural network technologies to optimise the control of the operating modes of a sucker-rod pump installation. Machinery & Energetics, 16 (1), 32–42. https://doi.org/10.31548/machinery/1.2025.32
5. Andriievskyi, I., Spivak, S., Gogota, O., Yermolenko, R. (2024). Application of the regression neural network for the analysis of the results of ultrasonic testing. Machinery & Energetics, 15 (1), 43–55. https://doi.org/10.31548/machinery/1.2024.43
6. Mukherjee, S., Claassen, M., Bürkner, P.-C. (2025). DGP-LVM: Derivative Gaussian process latent variable models. Statistics and Computing, 35 (5). https://doi.org/10.1007/s11222-025-10644-4

7. Shahan, J. T., Walker, S. W. (2025). Exact shape derivatives with unfitted finite element methods. Journal of Numerical Mathematics. https://doi.org/10.1515/jnma-2024-0113
8. Guo, P., Lan, Y., Qiao, J. (2025). Exact solutions of differential equations: renormalization group based polynomial scheme. Communications in Theoretical Physics, 77 (10), 105005. https://doi.org/10.1088/1572-9494/add24e
9. Konopatskiy, E. V., Bezditnyi, A. A. (2019). Geometric modeling and optimization of multidimensional data in Radischev integrated drawing. Journal of Physics: Conference Series, 1260 (7), 072006. https://doi.org/10.1088/1742-6596/1260/7/072006
10. Konopatskiy, E. V., Mashtaler, S. N., Bezditnyi, A. A. (2019). Study of high-strength steel fiber concrete strength characteristics under elevated temperatures using mathematical modelling methods. IOP Conference Series: Materials Science and Engineering, 687 (2), 022040. https://doi.org/10.1088/1757-899x/687/2/022040
11. Konopatskiy, E. V., Bezditnyi, A. A. (2020). Geometric modeling of multifactor processes and phenomena by the multidimensional parabolic interpolation method. Journal of Physics: Conference Series, 1441 (1), 012063. https://doi.org/10.1088/1742-6596/1441/1/012063
12. Lako, A., Barko, O. (2024). Design and optimisation of automated hydraulic gate control systems for flood control. Machinery & Energetics, 15 (4), 58–68. https://doi.org/10.31548/machinery/4.2024.58