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REACHABLE SETS OF A LAMÉ TYPE DYNAMICAL SYSTEM

M. I. Belishev *

V. A. Steklov Institute of Mathematics RAS
27, Fontanka, St. Petersburg 191011, Russia
belishev@pdmi.ras.ru

V. G. Fomenko

St. Petersburg State University
28, Universitetskii pr., Petrodvorets, St. Petersburg 198504, Russia
fomenkovova@mail.ru

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By a version of the boundary control method (*Belishev,* 1986)*, a Riemannian manifold is recovered via its dynamical boundary inverse data, which correspond to the scalar wave equation, with the help of the virtual sources. We extend this version to the dynamical vector Lamé-type system. Such an extension is based on studying the structure of the reachable sets. The prospective goal of our study is to solve the inverse problem of recovering parameters of the Lamé system from the dynamical boundary data. Bibliography*: 13 *titles. Illustrations*: 3 *figures.*

1 Introduction

In the case of the *scalar* wave equation, the problem of reconstructing a metric of a manifold from the boundary data was treated by using the version [1] of the boundary control method [2] with the help of the so-called virtual sources. From the physical point of view, the reconstruction procedure is reduced to localization of waves generated by the boundary sources (controls). The localization takes place in a small domain of a special shape, namely, a "cap" located near the end of the ray emanating at the boundary along the normal. Such a localization is possible due to an adequate choice of boundary controls [1, 3, 4].

The perspective goal of this paper is to extend the version [1] of the boundary control method to multi-velocity dynamical systems described by the *vector* wave equation. In such systems there are wave modes propagating with different velocities and interacting with each other, which makes the structure of reachable sets more complicated than that in the scalar case.

Let us describe the main result. We consider a Lamé type dynamical system with modes of

* To whom the correspondence should be addressed.

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two types (p-waves and s-waves) and restrict ourselves to the case of constant density ($\rho = 1$). The mode velocities c_p, c_s depend on the points and $c_p > c_s$ everywhere.

The main result of this paper concerns the following question. For a Lamé type system we reproduce all the steps of the "scalar" procedure [1] providing the localization of waves in the cap. What does such a procedure give us in our case?

The answer is as follows. We show that, in the vector case (a Lamé type system), there are two caps: one each at the ends of the p-ray and s-ray. In each cap, the corresponding mode is localized: potential fields in the p-cap and solenoidal fields in the s-cap. Such a separation of caps is an encouraging fact for the inverse problem of reconstructing the Lamé coefficients from the dynamical boundary data. Hopefully, this fact is also true for the full Lamé system. If this is the case, one can reconstruct coefficients by using an analog of the tools developed in [1, 3, 4] in a time-optimal way.

Let us emphasize the new points of the obtained result. The above-mentioned inverse problem for Lamé type systems was solved in [5]. In such systems, the wave field splits into potential and solenoidal components. This fact is essential for the approach of [5], where boundary controls are "sorted" into two classes so that controls of each class generate only p-waves or only s-waves respectively. In this paper, we avoid such a sorting, which can be regarded as a certain progress. The splitting of waves is still essential, but is used only for analyzing the structure of reachable sets and the action of the corresponding projections.

2 Geometry

2.1. Metric. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth[1)] boundary Γ. In $\overline{\Omega}$, a smooth function (*velocity*) $c = c(x) > 0$ is given. It determines in $\overline{\Omega}$ the conformal Euclidean metric

$$d\tau^2 := \frac{|dx|^2}{c^2}, \tag{2.1}$$

where $|dx|$ is the Euclidean element of length in $\mathbb{R}^3$. We denote by $\tau(x, y)$ the distance relative to this metric. We call $T^* := \max_{\Omega} \tau(\,\cdot\,\Gamma)$ the *filling time*.

For a subset $A \subset \overline{\Omega}$ we introduce its *metric neighborhoods*

$$\Omega^r[A] := \left\{x \in \overline{\Omega} \mid \tau(x, A) < r\right\}, \quad r > 0,$$

and denote by $\Omega^r := \Omega^r[\Gamma]$ the neighborhood of the boundary (the boundary layer of width r). The term "filling time" is motivated by the equality $T^* = \inf\{r > 0 \mid \Omega^r = \Omega\}$.

For $A \subset \overline{\Omega}$ we introduce the equidistant surface

$$\Gamma^r[A] := \left\{x \in \overline{\Omega} \mid \tau(x, A) = r\right\}, \quad r > 0$$

and denote by $\Gamma^r := \Gamma^r[\Gamma]$ the equidistance of the boundary.

With a point $x \in \overline{\Omega}$ we associate the set $\gamma(x) := \{\gamma \in \Gamma \mid \tau(x, \gamma) = \tau(x, \Gamma)\}$ of the nearest boundary points. It is known that for sufficiently small $r > 0$ and any $x \in \Omega^r$ the set $\gamma(x)$ consists of a single point and the system of semigeodesic (radial) coordinates with base Γ is

[1)] Throughout the paper, smooth surfaces, functions, fields etc. mean the smoothness of class C^∞.

regular in Ω^r. Let $T^{\rm reg}$ be the least upper bound of r for which such a regularity holds. The near-boundary layer $\Omega^{T^{\rm reg}}$ is called the *regular zone*.

2.2. Caps. Let $\sigma \subset \Gamma$ be a (small) closed subset with smooth boundary. For an example we mention the "disk" $D^r[\gamma] := \{\gamma' \in \Gamma \mid \tau(\gamma', \gamma) \leqslant r\}$ with small $r > 0$.

We fix positive $T < T^*$ and (small) $\varepsilon > 0$. By *caps* we mean sets of the form

$$\omega^{T,\varepsilon}[\sigma] := (\overline{\Omega^T} \backslash \Omega^{T-\varepsilon}) \cap \overline{\Omega^T[\sigma]}. \tag{2.2}$$

The typical shape of caps in the regular zone (for $T < T^{\rm reg}$) is illustrated by Figure 1. The dashed lines correspond to "vertical" rays (geodesic metrics (2.1) emanating from points of $\partial\sigma$ along the normal to Γ and directed towards the interior of Ω). The cap itself is shadowed.

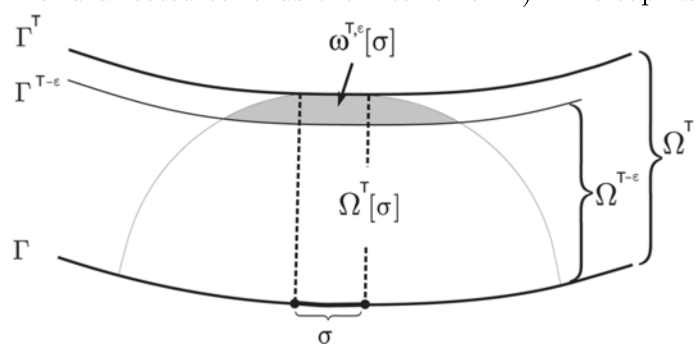

Fig. 1. A cap.

Caps are used for solving some inverse problems (cf. [1]–[4]). Their properties and behavior (with growth of T) are described in [2].

2.3. Domain of influence. In what follows, the variable $t \geqslant 0$ plays the role of time. We fix $T > 0$ and denote by $Q^T := \Omega \times (0, T)$ and $\Sigma^T := \Gamma \times [0, T]$ the space-time cylinder and its lateral surface respectively. For $(x_0, t_0) \in \overline{Q}^T = \overline{\Omega} \times [0, T]$ we introduce the *cone of influence* $K^T[(x_0, t_0)] := \{(x, t) \in \overline{Q}^T \mid \tau(x, x_0) \leqslant t - t_0\}$. For $B \subset \overline{Q}^T$ the subdomain

$$K^T[B] := \overline{\bigcup_{(x_0,t_0)\in B} K^T[(x_0, t_0)]}$$

is called the *domain of influence* of B. By this definition,

$$K^T[K^T[B]] = K^T[B]. \tag{2.3}$$

The smooth parts of the boundary of the domain of influence located inside Q^T are characteristic surfaces $\chi(x, t) = \text{const}$ defined by the known equations $\chi_t^2 - c^2 |\nabla\chi|^2 = 0$.

For $\sigma \subset \Gamma$ we set $\Sigma_\sigma^T := \overline{\sigma} \times [0, T]$. From the above definitions it follows that the cross-section $t = \xi$ of the domain of influence $K^T[\Sigma_\sigma^T]$ coincides with the ξ-neighborhood of σ in Ω

$$\{x \in \Omega \mid (x, \xi) \in K^T[\Sigma_\sigma^T]\} = \overline{\Omega}^\xi[\sigma], \quad 0 < \xi \leqslant T. \tag{2.4}$$

We consider the subset $\Xi^T[\sigma] := \Sigma^T \cap K^T[\Sigma_\sigma^T]$ of the lateral surface of the cylinder. By the above definitions and the property (2.3), we have

$$K^T[\Xi^T[\sigma]] = K^T[\Sigma_\sigma^T]. \tag{2.5}$$

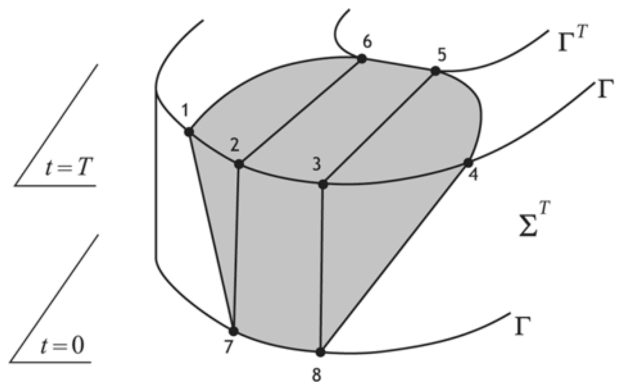

FIG. 2. The domains of influence.

In Figure 2, the set σ corresponds to the segment $\{7,8\}$, the part Σ^T_σ of the lateral surface Σ^T corresponds to the quadrangle $\{7,8,3,2\}$, the neighborhood $\Omega^T[\sigma]$ is bounded by the contour $\{1,2,3,4,5,6,1\}$, and the contour $\{1,7,8,4,3,2,1\}$ bounds the set $\Xi^T[\sigma]$ on Σ^T.

2.4. A pair of metrics. With the Lamé system we associate the pair of metrics of the form (2.1) defined by the velocities of the wave modes c_p and c_s, where $c_p > c_s$ everywhere in $\overline{\Omega}$. To distinguish the metrics, we use the subscripts p and s for all the above-introduced objects: metrics, domains, spaces (below), operators and so on.

By the relation for velocities, $\tau_p(x,y) < \tau_s(x,y)$ and $\Omega^r_p[A] \supset \Omega^r_s[A]$ for any $x,y \in \overline{\Omega}$ $(x \neq y)$, $A \subset \overline{\Omega}$, and $r > 0$. We set $T^{\rm reg} := \min\{T^{\rm reg}_p, T^{\rm reg}_s\}$ and $\Omega^{T^{\rm reg}} := \Omega^{T^{\rm reg}}_p$.

The picture of neighborhoods in the regular zone is presented in Figure 3(a). The caps $\omega^{T,\varepsilon}_p[\sigma]$ and $\omega^{T,\varepsilon}_s[\sigma]$ are shadowed. The pair of metrics determines the subdomain (bounded by the contour $\{1,2,3,4,5,6,7,8,9,10,1\}$ in Figure 3(a)) in Ω of the form

$$\Lambda^T_{ps}[\sigma] := \big\{x \in \Omega \mid (x,T) \in K^T_s[\Xi^T_p[\sigma]]\big\} \supset \Omega^T_s[\sigma]. \tag{2.6}$$

Its part $\Lambda^T_{ps}[\sigma]\backslash\Omega^T_s[\sigma]$ in dynamics corresponds to the zone (hatched), where there are the so-called *lateral waves*. For sufficiently small ε this part is at a positive distance to the cap $\omega^{T,\varepsilon}_s[\sigma]$.

Figure 3(b) illustrates the geometry of domains of influence for the pair of metrics. The domains $\Xi^T_p[\sigma]$ and $\Xi^T_s[\sigma]$ are bounded by the contours $\{1,7,8,6,5,4,3,2,1\}$ and $\{2,7,8,5,4,3,2\}$ respectively.

2.5. Functions and fields. We consider the following sets of real-valued scalar and vector ($\mathbb{R}^3$-valued) functions. The latter are called *fields*.

2.5.1. Space $\mathscr{H}$. The main role is played by the space of fields $\mathscr{H} := L_2(\Omega;\mathbb{R}^3)$ equipped with the inner product

$$(y,v)_{\mathscr{H}} = \int\limits_\Omega y(x)\cdot v(x)\,dx,$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^3$. For measurable sets $A \subset \Omega$ we introduce the subspaces $\mathscr{H}[A] := \big\{y \in \mathscr{H} \mid \operatorname{supp} y \subset \overline{A}\big\}$. In the space $\mathscr{H}$, we extract the (sub)spaces $\mathscr{G} := \big\{y = \nabla\varphi \mid \varphi \in W^1_2(\Omega)\big\}$ and $\mathscr{J} := \{y \in \mathscr{H} \mid \operatorname{div} y = 0\}$ of potential and solenoidal fields [6, 7]. Their subspaces of fields located in A are denoted by $\mathscr{G}[A]$ and $\mathscr{J}[A]$.

2.5.2. Space $\mathscr{F}^T$. We introduce the space $\mathscr{F}^T := L_2(\Sigma^T;\mathbb{R}^3)$ equipped with the inner product

$$(f,g)_{\mathscr{F}^T} := \int\limits_{\Sigma^T} f(\gamma,t)\cdot g(\gamma,t)\,d\Gamma\,dt,$$

where $d\Gamma$ is the Euclidean element of area on Γ.

The class $\dot{\mathscr{F}}^T := \{f \in C^\infty(\Sigma^T;\mathbb{R}^3) \mid \operatorname{supp} f \subset \Gamma\times(0,T]\}$ of smooth fields is dense in $\mathscr{F}^T$. We note that the fields in $\dot{\mathscr{F}}^T$ vanish in a neighborhood of $t=0$.

With a subset $B \subset \Sigma^T$ we associate the subspace $\mathscr{F}^T[B] := \{f \in \mathscr{F}^T \mid \operatorname{supp} f \subset \overline{B}\}$. It contains the dense set $\dot{\mathscr{F}}^T[B] := \mathscr{F}^T[B] \cap \dot{\mathscr{F}}^T$ of smooth fields.

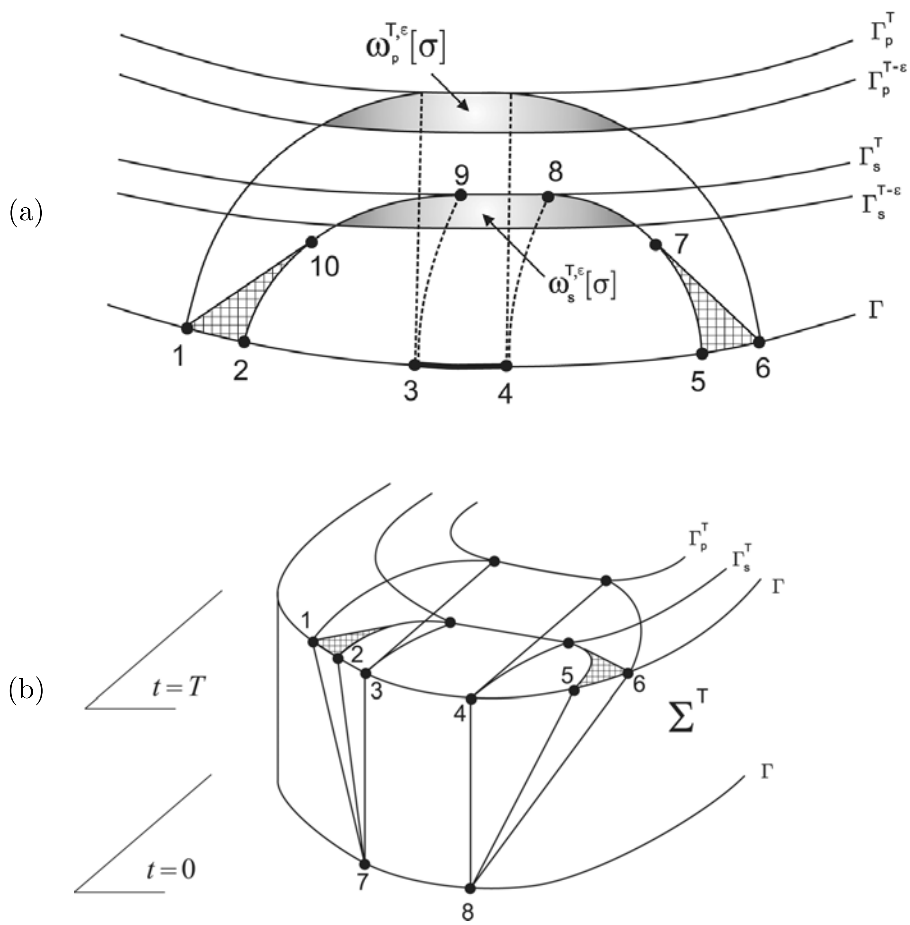

FIG. 3. A pair of metrics.

A vector $a \in \mathbb{R}^3$ at a boundary point is decomposed into the sum

$$a = a_\nu + a_\theta = a^\nu\,\nu + a_\theta, \tag{2.7}$$

where ν is the Euclidean outward unit normal to Γ, $a^\nu = a\cdot\nu$, and a_ν and a_θ are the normal and tangent components. With this decomposition we associate the tangent component and write $a = \begin{pmatrix} a^\nu \\ a_\theta \end{pmatrix}$.

We introduce the scalar and vector spaces $\mathscr{F}_p^T := L_2(\Sigma^T)$, $\mathscr{F}_s^T := \{f \in \mathscr{F}^T \mid \nu \cdot f\big|_\Gamma = 0\}$. Their subspaces $\mathscr{F}_\lambda^T[B]$ $(\lambda = p, s)$ consist of elements supported in $\overline{B}$, $\dot{\mathscr{F}}_\lambda^T[B]$ are smooth functions and fields vanishing in a neighborhood of $t = 0$. In accordance with (2.7), we have

$$\mathscr{F}^T = \begin{pmatrix} \mathscr{F}_p^T \\ \mathscr{F}_s^T \end{pmatrix}.$$

3 The Lamé System

3.1. The initial-boundary value problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. We fix $T \in (0, \infty)$.

The dynamical *Lamé system* governs the propagation of waves in an elastic medium occupying Ω. We write the system in the invariant (coordinate-free) form

$$\rho u_{tt} = Lu \quad \text{in } Q^T, \tag{3.1}$$

$$u\big|_{t=0} = u_t\big|_{t=0} = 0 \quad \text{in } \overline{\Omega}, \tag{3.2}$$

$$u = f \quad \text{on } \Sigma^T, \tag{3.3}$$

where

$$L := \nabla(\lambda + 2\mu)\,\mathrm{div} \, - \mathrm{rot}\,\mu\,\mathrm{rot} \, + 2\left([\nabla\mu \times \mathrm{rot}\,] + [\nabla\mu \times \mathrm{rot}\,]^* + H_\mu - q\right) \tag{3.4}$$

is the operator with smooth coefficients $\rho(x) > 0$, $\mu(x) > 0$, $3\lambda + 2\mu > 0$ in $\overline{\Omega}$, $[\ldots]^*$ denotes the conjugation in the sense of Lagrange, H_μ is the operator of multiplication by the matrix-valued function of second order derivatives $(H_\mu)_{ik} = \dfrac{\partial^2 \mu}{\partial x_i \partial x_k}$, q is the operator of multiplication by $q = \Delta\mu$ (cf. [8]).

The $\mathbb{R}^3$-valued function $f = f(\gamma, t)$ is called the (Dirichlet) *boundary control*. It describes displacements of boundary points generating the wave process in Ω. The solution $u = u^f(x, t)$ (*wave*) is an $\mathbb{R}^3$-valued function describing displacements of points of the medium in Ω. For controls of class $\dot{\mathscr{F}}^T$ the problem (3.1)–(3.3) has a unique classical smooth solution u^f.

The mapping $f \mapsto u^f$ is continuous from $\mathscr{F}^T$ to $L_2\left((0, T); L_2(\Omega; \mathbb{R}^3)\right)$ (cf. [9]). Consequently, it can be extended from $\dot{\mathscr{F}}^T$ to controls of $\mathscr{F}^T$ by continuity. By a (weak) solution to the problem (3.1)–(3.3) for controls of this class we mean the image of f under the action of this extension.

3.2. The finiteness of the domain of influence. The functions

$$c_p := \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \quad c_s := \left(\frac{\mu}{\rho}\right)^{1/2}$$

$(c_p > c_s)$ are interpreted as velocities of the longitudinal (fast) and transversal (slow) modes respectively. The velocities determine two conformal Euclidean metrics

$$d\tau_\lambda^2 := \frac{|dx|^2}{c_\lambda^2} \quad (\lambda = p, s)$$

in $\overline{\Omega}$. Each of them determines distances, neighborhoods, geodesics, domains of influence etc.

The Lamé equation is hyperbolic and possesses two families of characteristics $\chi(x, t) = \mathrm{const}$ in Q^T defined by the equations $(\chi_t)^2 - c_\lambda^2 |\nabla\chi|^2 = 0$.

By the hyperbolicity of the problem (3.1)–(3.3), we have the relation

$$\operatorname{supp} u^f \subset K_p^T[\operatorname{supp} f], \tag{3.5}$$

known as the *finiteness principle for domain of influence*, which shows that the waves of the Lamé system propagate with velocity less than or equal to the velocity of the fast mode c_p.

Let $\sigma \subset \Gamma$, and let $f \in \mathscr{F}^T[\Sigma_\sigma^T]$, i.e., the control f acts with σ. Taking into account (2.4), from (3.5) we find

$$\operatorname{supp} u^f(\,\cdot\,t) \subset \overline{\Omega_p^t[\sigma]}, \quad t > 0. \tag{3.6}$$

3.3. System α^T. Hereinafter, we regard the problem (3.1)–(3.3) as a dynamical system, denoted by α^T, and equip it with attributes of control theory, spaces and operators.

The space of controls $\mathscr{F}^T$ is called the *outer space* of the system α^T. The solution u^f is interpreted as the *trajectory* of the system, and $u^f(\,\cdot\,t)$ is the *state* of the system at time t. The space $\mathscr{H} := L_{2,\rho}(\Omega;\mathbb{R}^3)$ equipped with the inner product [2)]

$$(u,v)_{\mathscr{H}} = \int\limits_\Omega u(x)\cdot v(x)\,\rho(x)dx$$

is referred to the *inner space* of the system α^T. By the L_2-regularity of the solution (cf. Subsection 3.1), all waves $u^f(\,\cdot\,t)$ belong to this space.

By (3.6), the relation $f \in \mathscr{F}^T[\Sigma_\sigma^T]$ implies $u^f(\,\cdot\,t) \in \mathscr{H}[\Omega_p^t[\sigma]]$ for all $0 < t \leqslant T$, i.e., the entire trajectory u^f of the system α^T does not leave the subspaces $\mathscr{H}[\Omega_p^T[\sigma]]$.

3.4. Controllability. In the system α^T, the set of states (waves)

$$\mathscr{U}[\Sigma_\sigma^T] := \{u^f(\,\cdot\,T) \mid f \in \dot{\mathscr{F}}^T[\Sigma_\sigma^T]\}$$

is said to be *reachable* (from a part σ of the boundary in time $t = T$). By (3.6),

$$\mathscr{U}[\Sigma_\sigma^T] \subset \mathscr{H}[\Omega_p^T[\sigma]], \quad \sigma \subseteq \Gamma, \quad T > 0. \tag{3.7}$$

The properties of reachable sets and the characteristic of embeddings (3.7) are central aspects of the boundary control theory. We formulate a result established in [9] on the basis of the fundamental theorem about the uniqueness of an extension of the solution to the Lamé equation through a noncharacteristic surface [10].

Let $X_{\Omega_s^T[\sigma]}$ be an (orthogonal) projection in $\mathscr{H}$ onto $\mathscr{H}[\Omega_s^T[\sigma]]$ such that $\Omega_s^T[\sigma]$:

$$X_{\Omega_s^T[\sigma]}\,y = \begin{cases} y & \text{in } \Omega_s^T[\sigma], \\ 0 & \text{in } \Omega\backslash\Omega_s^T[\sigma]. \end{cases}. \tag{3.8}$$

We have

$$\overline{X_{\Omega_s^T[\sigma]}\,\mathscr{U}[\Sigma_\sigma^T]} = \mathscr{H}[\Omega_s^T[\sigma]], \quad \sigma \subseteq \Gamma, \quad T > 0 \tag{3.9}$$

(the closure is taken in the metric of $\mathscr{H}$).

[2)] We use the same symbol $\mathscr{H}$ as in Subsection 2.5 for the sake of simplicity. Below, we assume that $\rho = 1$ in the *Lamé type system.* The notation $\mathscr{H}[A]$ has the same sense as in Subsection 2.5.

From (3.9) it follows that any vector field $y \in L_{2,\rho}(\Omega_s^T[\sigma]; \mathbb{R}^3)$ that is localized in a subdomain covered by the slow mode, can be approximated (with any accuracy) by the wave $u^f(\,\cdot\,T)$ under an appropriate choice of the control $f \in \dot{\mathscr{F}}^T[\Sigma_\sigma^T]$. In control theory, this property is known as the *local approximate boundary controllability* of the system α^T.

At final time $t = T$, the waves generated by controls $f \in \mathscr{F}^T[\Sigma_\sigma^T]$ occupy the "fast" domain $\Omega_p^T[\sigma]$ containing the "slow" subdomain $\Omega_s^T[\sigma]$. The relation (3.9), roughly speaking, means that the shape of the wave $u^f(\,\cdot\,T)$ in $\Omega_s^T[\sigma]$ can be arbitrary. At the same time, this is not the case in the subdomain $\Omega_p^T[\sigma]\backslash\Omega_s^T[\sigma]$. No effective description of parts $u^f(\,\cdot\,T)$ in $\Omega_p^T[\sigma]\backslash\Omega_s^T[\sigma]$ is known yet, which causes certain difficulties for studying the inverse problem (cf. [2]) and forces us to consider a simplified model of the system (3.1)–(3.3), namely, a *Lamé type system*. The required description will be obtained and used within the framework of this model.

4 System of Lamé Type

4.1. System α^T. Keeping the higher-order (with respect to differentiation) terms in (3.1)–(3.4) and setting $\rho = 1$, we obtain the *Lamé type system*

$$u_{tt} = \nabla \varkappa \operatorname{div} u - \operatorname{rot} \mu \operatorname{rot} u \quad \text{in} \;\; Q^T, \tag{4.1}$$

$$u\big|_{t=0} = u_t\big|_{t=0} = 0 \quad \text{in} \;\; \overline{\Omega}, \tag{4.2}$$

$$u = f \quad \text{on} \;\; \Sigma^T, \tag{4.3}$$

where $\varkappa := \lambda + 2\mu$. This system will be still denoted by α^T.

It is easy to show that the problem (4.1)–(4.3) has the same regularity properties as the problem (3.1)–(3.3). Furthermore, like (3.5),

$$\operatorname{supp} u^f \subset K_p^T[\operatorname{supp} f]. \tag{4.4}$$

All attributes of a dynamical system, i.e., outer and inner spaces, their subspaces etc. remain the same for systems of Lamé type. In this case, the role of inner space is played by $\mathscr{H} = L_2(\Omega; \mathbb{R}^3)$. The character of controllability is the same: the relation (3.9) remains valid. However, unlike the general case, two subsystems of the Lamé type system can be naturally distinguished: *acoustic* and *Maxwell* ones.

4.2. Subsystem α_p^T. We consider the *scalar* initial-boundary value problem

$$\varphi_{tt} = c_p^2 \Delta \varphi \quad \text{in} \;\; Q^T, \tag{4.5}$$

$$\varphi\big|_{t=0} = \varphi_t\big|_{t=0} = 0 \quad \text{in} \;\; \overline{\Omega}, \tag{4.6}$$

$$\varphi = g \quad \text{on} \;\; \Sigma^T \tag{4.7}$$

where $c_p := \sqrt{\varkappa}$. For controls of class $\dot{\mathscr{F}}_p^T$ this problem has a unique classical smooth solution $\varphi = \varphi^g(x,t)$. The mapping $g \mapsto \varphi^g$ is continuous from $\mathscr{F}^T$ to $L_2((0,T); L_2(\Omega))$, which allows us to introduce a solution for $g \in \mathscr{F}_p^T$. The corresponding dynamical system is said to be *acoustic* and is denoted by α_p^T. The outer and inner spaces are $\mathscr{F}_p^T$ and $\mathscr{H}_p := L_2(\Omega)$ respectively.

Since the domain of influence is finite for the wave equation (4.5), we have the relation

$$\operatorname{supp} \varphi^g \subset K_p^T[\operatorname{supp} g] \tag{4.8}$$

and its consequence

$$\operatorname{supp}\varphi^g(\,\cdot\,t)\,\subset\,\overline{\Omega_p^t[\sigma]},\quad t>0, \tag{4.9}$$

for controls g acting with $\sigma \subseteq \Gamma$.

The acoustic system is locally approximately controllable from the boundary. We define the reachable sets

$$\Phi[\Sigma_\sigma^T] := \left\{\varphi^g(\,\cdot\,T) \mid g \in \dot{\mathscr{F}}_p^T[\Sigma_\sigma^T]\right\}. \tag{4.10}$$

Using the Holmgren–John–Tataru theorem (cf. [2, 11, 12]), it is possible to prove that

$$\overline{\Phi[\Sigma_\sigma^T]} = \mathscr{H}_p[\Omega_p^T[\sigma]] \tag{4.11}$$

(the closure is taken in $\mathscr{H}$) for all $\sigma \subseteq \Gamma$ and $T > 0$.

We indicate a consequence of (4.11) which will be used in Section 5 below. Denote

$$\nabla\Phi[\Sigma_\sigma^T] := \left\{\nabla\varphi^g(\,\cdot\,T) \mid g \in \dot{\mathscr{F}}_p^T[\Sigma_\sigma^T]\right\}.$$

Let $T < T^{\mathrm{reg}}$. In this case, the neighborhood $\Omega_p^T[\sigma]$ is bounded by a piecewise smooth surface. We have

$$\overline{\nabla\Phi[\Sigma_\sigma^T]} = \left\{\nabla q \mid q \in W_2^1(\Omega),\, \operatorname{supp} q \subset \overline{\Omega_p^T[\sigma]},\, q|_{\Gamma\setminus\sigma} = 0\right\} \tag{4.12}$$

(the closure is taken in $\mathscr{H}$). This means the completeness of the gradients of waves in the space of potential fields localized in $\Omega_p^T[\sigma]$.

Regarding the derivation of equality, we note the following. By (4.9) (with $t = T$) and the condition supp $g(\,\cdot\,T) \subset \overline{\sigma}$ following from $\operatorname{supp} g \subset \Sigma_\sigma^T$, the wave $\varphi^g(\,\cdot\,T)$ satisfies all the conditions on q on the right-hand side of (4.12). Consequently, the left-hand side is embedded into the right-hand side. Therefore, the violation of (4.12) means the existence of ∇q that is orthogonal to all $\nabla\varphi^g(\,\cdot\,T)$. Then for sufficiently smooth q

$$\begin{aligned}
0 = (\nabla q, \nabla\varphi^g(\,\cdot\,T))_{\mathscr{H}} &= \int\limits_{\Omega_p^T[\sigma]} \nabla q\cdot\nabla\varphi^g(\,\cdot\,T)\,dx \\
&= \int\limits_{\partial\Omega_p^T[\sigma]} \frac{\partial q}{\partial\nu}\,\varphi^g(\,\cdot\,T)\,d\Gamma - \int\limits_{\Omega_p^T[\sigma]} \Delta q\,\varphi^g(\,\cdot\,T)\,dx \overset{(4.7),(4.9)}{=} \\
&= \int\limits_{\sigma} \frac{\partial q}{\partial\nu}\,g(\,\cdot\,T)\,d\Gamma - \int\limits_{\Omega_p^T[\sigma]} \Delta q\,\varphi^g(\,\cdot\,T)\,dx.
\end{aligned} \tag{4.13}$$

The controls $g \in \dot{\mathscr{F}}_p^T[\Sigma_\sigma^T]$ such that $g(\,\cdot\,T) = 0$ are dense in $\mathscr{F}_p^T[\Sigma_\sigma^T]$. The corresponding waves $\varphi^g(\,\cdot\,T)$ are dense in $\mathscr{H}_p[\Omega_p^T[\sigma]]$. Substituting such a control g into (4.13), we see that $\Delta q = 0$ in $\Omega_p^T[\sigma]$. Returning to an arbitrary control $g \in \dot{\mathscr{F}}_p^T[\Sigma_\sigma^T]$ in (4.13), we get

$$0 = \int\limits_{\sigma} \frac{\partial q}{\partial\nu}\,g(\,\cdot\,T)\,d\Gamma$$

and, consequently, $\left.\dfrac{\partial q}{\partial\nu}\right|_\sigma = 0$. Thus, q satisfies the conditions

$$\Delta q = 0 \ \text{ in } \Omega_p^T[\sigma], \quad q\big|_{(\partial\Omega_p^T[\sigma])\setminus\sigma} = 0, \quad \left.\frac{\partial q}{\partial\nu}\right|_\sigma = 0.$$

It is clear that $q = 0$. The case of a nonsmooth control q is reduced to the above case by an appropriate regularization (cf. [2, AppendixA2]).

To complete the consideration of α_p^T, we give an auxiliary result. We extend the reachable set (4.10) by passing to controls localized on $\Xi_p^T[\sigma] \supset \Sigma_\sigma^T$ (cf. Figures 2 and 3):

$$\Phi[\Xi_p^T[\sigma]] := \left\{ \varphi^g(\,\cdot\, T) \mid g \in \dot{\mathscr{F}}_p^T[\Xi_p^T[\sigma]] \right\} \supset \Phi[\Sigma_\sigma^T]. \tag{4.14}$$

By (2.5), (4.8), and (4.9), the trajectories φ^g with controls $g \in \dot{\mathscr{F}}_p^T[\Xi_p^T[\sigma]]$ do not leave the neighborhood $\Omega_p^T[\sigma]$. As a consequence of (4.11) and (4.12), we obtain the equalities

$$\overline{\Phi[\Xi_p^T[\sigma]]} = \mathscr{H}_p[\Omega_p^T[\sigma]], \overline{\nabla\Phi[\Xi_p^T[\sigma]]} = \left\{ \nabla q \mid q \in W_2^1(\Omega),\, \operatorname{supp} q \subset \overline{\Omega_p^T[\sigma]},\, q|_{\Gamma\setminus\sigma} = 0 \right\}. \tag{4.15}$$

If $\sigma = \Gamma$, then $\Xi_p^T[\sigma] = \Sigma^T$, the boundary condition in (4.15) is eliminated, and we get

$$\overline{\nabla\Phi[\Sigma^T]} = \left\{ \nabla q \mid q \in W_2^1(\Omega),\, \operatorname{supp} q \subset \overline{\Omega_p^T} \right\} = \mathscr{G}[\Omega_p^T]. \tag{4.16}$$

4.3. Subsystem α_s^T. We consider the vector initial-boundary value problem

$$\psi_{tt} = -c_s^2 \operatorname{rot}\operatorname{rot}\psi \quad \text{in } \; Q^T, \tag{4.17}$$

$$\psi\big|_{t=0} = \psi_t\big|_{t=0} = 0 \quad \text{in } \; \overline{\Omega}, \tag{4.18}$$

$$\psi_\theta \times \nu = h \quad \text{on } \; \Sigma^T \tag{4.19}$$

where $c_s := \sqrt{\mu} < c_p$, ψ_θ is the tangent component of ψ (cf. (2.7)), and the symbol $\times$ denotes the vector product in $\mathbb{R}^3$. For $h \in \dot{\mathscr{F}}_s^T$ the problem has a unique classical smooth solution $\psi = \psi^h(x,t)$. We note that the mapping $h \mapsto \psi^h$ defined on the smooth class $\dot{\mathscr{F}}_s^T$ is not continuous from $\mathscr{F}_s^T$ to $L^2((0,T); L_2(\Omega;\mathbb{R}^3))$ (cf. [13]). However, this difficulty is technical, so that the further consideration will deal with smooth controls and solutions.

The corresponding dynamical system is called the *Maxwell system* and is denoted by α_s^T. The outer space for this system is $\mathscr{F}_s^T$. It is convenient to take $\mathscr{H}$ for the inner space, but the following remark is essential. The quantity $\operatorname{div}\psi^h$ is the motion integral for the system α_s^T and $\operatorname{div}\psi^h(\,\cdot\, t) = 0$ for all $t \geqslant 0$ in view of the initial condition (4.18). Therefore, the waves are solenoidal fields and the trajectory of the system lies in the subspace $\mathscr{J}$ (cf. Subsection 2.5).

Equation (4.17) is obtained from the full Maxwell system by excluding one of the components (magnetic field). Since the domain of influence for the Maxwell equation is finite, we have the relation

$$\operatorname{supp}\psi^h \subset K_s^T[\operatorname{supp} h] \tag{4.20}$$

and its consequence

$$\operatorname{supp}\psi^h(\,\cdot\, t) \subset \overline{\Omega_s^t[\sigma]}, \quad t > 0, \tag{4.21}$$

for controls h acting with $\sigma \subseteq \Gamma$.

The system α_s^T is locally approximately controllable from the boundary in the following sense. We define the reachable sets by the formula

$$\Psi[\Sigma_\sigma^T] := \left\{ \psi^h(\,\cdot\, T) \mid h \in \dot{\mathscr{F}}_s^T[\Sigma_\sigma^T] \right\} \tag{4.22}$$

and introduce the subspace $\mathscr{J}[\Omega_s^T[\sigma]] := \{y \in \mathscr{J} \mid \operatorname{supp} y \subset \overline{\Omega_s^T[\sigma]}\}$. By the uniqueness of an extension of the solution to the Maxwell equations through a noncharacteristic surface [10], one can prove that

$$\overline{\Psi[\Sigma_\sigma^T]} = \mathscr{J}[\Omega_s^T[\sigma]] \tag{4.23}$$

(the closure is taken in $\mathscr{H}$) for all $\sigma \subseteq \Gamma$ and $T > 0$ (cf. [2, Theorem 3]).

We formulate a consequence of (4.23) which will be used in Section 5 below. Denote

$$\operatorname{rot} \Psi[\Sigma_\sigma^T] := \left\{ \operatorname{rot} \psi^h(\,\cdot\,T) \mid h \in \dot{\mathscr{F}}_s^T[\Sigma_\sigma^T] \right\}.$$

Let the set σ be simply connected on Γ. In this case, for $T < T^{\mathrm{reg}}$ the neighborhood $\Omega_s^T[\sigma]$ is bounded by a piecewise smooth surface and is topologically homeomorphic to the Euclidean half-ball. Then

$$\overline{\operatorname{rot} \Psi[\Sigma_\sigma^T]} = \left\{ \operatorname{rot} a \mid a \in W_2^1(\Omega; \mathbb{R}^3),\ \operatorname{supp} a \subset \overline{\Omega_s^T[\sigma]},\ \nu \times a|_{\Gamma\setminus\sigma} = 0 \right\} = \mathscr{J}[\Omega_s^T[\sigma]]. \tag{4.24}$$

To prove the first equality, one can use the scheme of the proof of (4.11) (integration by parts) and the equality (4.23). We omit details. The second equality in (4.24) is a consequence of fact that the curls of smooth fields are dense in $\mathscr{J}[\Omega_s^T[\sigma]]$ (cf. [7]). We note that the shape of the neighborhood $\Omega_s^T[\sigma]$ plays an essential role: since it is homeomorphic to a half-ball, the solenoidal fields in $\mathscr{J}[\Omega_s^T[\sigma]]$ have vector potentials (i.e., are curls).

To complete the consideration of α_s^T, we give the following auxiliary result. We extend the reachable set (4.22) by passing to the controls localized on $\Xi_p^T[\sigma] \supset \Sigma_\sigma^T$:

$$\Psi[\Xi_p^T[\sigma]] := \left\{ \psi^h(\,\cdot\,T) \mid h \in \dot{\mathscr{F}}_s^T[\Xi_p^T[\sigma]] \right\} \supset \Psi[\Sigma_\sigma^T]. \tag{4.25}$$

By (4.20), the trajectories ψ^h with controls $h \in \dot{\mathscr{F}}_s^T[\Xi_p^T[\sigma]]$ do not leave the subdomain $\Lambda_{ps}^T[\sigma] \supset \Omega_s^T[\sigma]$ (cf. (2.6)). From (4.23)–(4.25) it follows that

$$\overline{\Psi[\Xi_p^T[\sigma]]} \supset \mathscr{J}[\Omega_s^T[\sigma]], \quad \overline{\operatorname{rot} \Psi[\Xi_p^T[\sigma]]} \supset \mathscr{J}[\Omega_s^T[\sigma]]. \tag{4.26}$$

If $\sigma = \Gamma$, then $\Xi_p^T[\sigma] = \Sigma^T$, the boundary condition in (4.24) is eliminated, and we get

$$\overline{\Psi[\Sigma^T]} = \overline{\operatorname{rot} \Psi[\Sigma^T]} = \mathscr{J}[\Omega_s^T]. \tag{4.27}$$

4.4. Connection between trajectories. We choose a control $f \in \dot{\mathscr{F}}^T$ in the system (4.1)–(4.3) and set

$$g := \left[\varkappa \operatorname{div} u^f\right]\big|_{\Sigma^T} \in \dot{\mathscr{F}}_p^T, \quad h := \left[\mu\, (\operatorname{rot} u^f)_\theta \times \nu\right]\big|_{\Sigma^T} \in \dot{\mathscr{F}}_s^T. \tag{4.28}$$

As was shown in [5], the following representation holds:

$$u^f = \nabla \varphi^g + \operatorname{rot} \psi^h \quad \text{in } Q^T, \tag{4.29}$$

which connects the trajectories of the system α^T with the trajectories of its subsystems α_p^T and α_s^T. This means that the waves of the Lamé type system split into potential and solenoidal components.

On the other hand, for arbitrary $g \in \dot{\mathscr{F}}_p^T$ and $h \in \dot{\mathscr{F}}_s^T$ the fields $\nabla\varphi^g = u^{f'}$ and $\operatorname{rot}\psi^h = u^{f''}$ are trajectories of the system α^T corresponding to the controls

$$f' = \begin{pmatrix} \nu\cdot\nabla\varphi^g|_{\Sigma^T} \\ (\nabla\varphi^g)_\theta|_{\Sigma^T} \end{pmatrix}, \quad f'' = \begin{pmatrix} \nu\cdot\operatorname{rot}\psi^h|_{\Sigma^T} \\ (\operatorname{rot}\psi^h)_\theta|_{\Sigma^T}. \end{pmatrix} \tag{4.30}$$

Therefore, $\nabla\varphi^g + \operatorname{rot}\psi^h = u^{f'} + u^{f''} = u^{f'+f''}$. Taking into account (4.29), we obtain the representation in the form of an algebraic sum

$$\mathscr{U}[\Sigma^T] = \nabla\Phi[\Sigma^T] + \operatorname{rot}\Psi[\Sigma^T]. \tag{4.31}$$

Using (4.16) and (4.27) and passing to the closure, it is easy to find that

$$\overline{\mathscr{U}[\Sigma^T]} = \mathscr{G}[\Omega_p^T] + \mathscr{J}[\Omega_s^T]. \tag{4.32}$$

Note that the terms of this sum have nonzero intersection.

The representation (4.31) can be specified for controls acting from a part of the boundary as follows:

$$\mathscr{U}[\Xi_p^T[\sigma]] = \nabla\Phi[\Xi_p^T[\sigma]] + \operatorname{rot}\Psi[\Xi_p^T[\sigma]]. \tag{4.33}$$

Indeed, let $f \in \dot{\mathscr{F}}^T[\Xi_p^T[\sigma]]$ be such that $u^f(\,\cdot\,T) \in \mathscr{U}[\Xi_p^T[\sigma]]$. By (4.4), for the controls g and h defined in (4.28) we have $g \in \dot{\mathscr{F}}_p^T[\Xi_p^T[\sigma]]$ and $h \in \dot{\mathscr{F}}_s^T[\Xi_p^T[\sigma]]$. From (4.29) it follows that $u^f(\,\cdot\,T) = \nabla\varphi^g(\,\cdot\,T) + \operatorname{rot}\psi^h(\,\cdot\,T)$, where $\nabla\varphi^g(\,\cdot\,T) \in \nabla\Phi[\Xi_p^T[\sigma]]$ and $\operatorname{rot}\psi^h(\,\cdot\,T) \in \operatorname{rot}\Psi[\Xi_p^T[\sigma]]$. Conversely, assume that $g \in \dot{\mathscr{F}}_p^T[\Xi_p^T[\sigma]]$ and $h \in \dot{\mathscr{F}}_s^T[\Xi_p^T[\sigma]]$. Then $f', f'', f'+f'' \in \dot{\mathscr{F}}^T[\Xi_p^T[\sigma]]$ in (4.30) and $\nabla\varphi^g(\,\cdot\,T) + \operatorname{rot}\psi^h(\,\cdot\,T) = u^{f'+f''}(\,\cdot\,T)$ belongs to the reachable set $\mathscr{U}[\Xi_p^T[\sigma]]$. Thus, (4.31) is a particular case of the representation (4.33) for $\sigma = \Gamma$.

5 Separation of Caps

We fix positive $T < T^{\text{reg}}$. Let $\sigma \subset \Gamma$ be a closed simply connected set with piecewise smooth boundary. We recall that caps are defined in (2.2). We choose a small positive number ε such that $(\Lambda_{ps}^T[\sigma]\backslash\Omega_s^T[\sigma]) \cap \omega_s^{T,\varepsilon}[\sigma] = \varnothing$ (cf. (2.6)).

5.1. Caps in the subsystems. We return to the acoustic subsystem α_p^T and consider the reachable sets

$$\begin{aligned} \Phi^T &:= \Phi[\Sigma^T] = \left\{\varphi^g(\,\cdot\,T) \mid g \in \dot{\mathscr{F}}_p^T[\Sigma^T]\right\}, \\ \Phi^{T-\varepsilon} &:= \Phi[\Gamma\times[\varepsilon,T]] = \left\{\varphi^g(\,\cdot\,T) \mid g \in \dot{\mathscr{F}}_p^T[\Sigma^T],\ g\big|_{\Gamma\times[0,\varepsilon]} = 0\right\}, \\ \Phi[\Xi_p^T[\sigma]] &= \left\{\varphi^g(\,\cdot\,T) \mid g \in \dot{\mathscr{F}}_p^T[\Xi_p^T[\sigma]]\right\}. \end{aligned}$$

The set $\Phi^{T-\varepsilon}$ is formed by the waves generated by retarded controls. By (4.8) and (4.9), the waves in $\Phi^{T-\varepsilon}$ are localized in $\Omega_p^{T-\varepsilon}$. Since the operator L defining the evolution of the system α^T and its subsystems is independent of time, the properties of $\Phi^{T-\varepsilon}$ are similar to those of Φ^T.

By the controllabitily of (4.15) (in particular, for $\sigma = \Gamma$), we have $\overline{\Phi^T} = \mathscr{H}_p[\Omega_p^T]$, $\overline{\Phi^{T-\varepsilon}} = \mathscr{H}_p[\Omega_p^{T-\varepsilon}]$, and $\overline{\Phi[\Xi_p^T[\sigma]]} = \mathscr{H}_p[\Omega_p^T[\sigma]]$. The subspaces $\mathscr{H}_p[\dots]$ consist of functions localized in the corresponding subdomains of Ω. The projections onto these subspaces act like cut-off functions (cf. (3.8)). As a consequence,

$$(\overline{\Phi^T} \ominus \overline{\Phi^{T-\varepsilon}}) \cap \overline{\Phi[\Xi_p^T[\sigma]]} = \mathscr{H}_p\left[\left(\Omega_p^T\backslash\Omega_p^{T-\varepsilon}\right) \cap \Omega_p^T[\sigma]\right].$$

By (2.2), we have

$$(\overline{\Phi^T} \ominus \overline{\Phi^{T-\varepsilon}}) \cap \overline{\Phi[\Xi_p^T[\sigma]]} = \mathscr{H}_p[\omega_p^{T,\varepsilon}]. \tag{5.1}$$

For the Maxwell subsystem α_s^T we consider the reachable sets

$$\Psi^T := \Psi[\Sigma^T] = \left\{\psi^h(\,\cdot\,T) \mid h \in \dot{\mathscr{F}}_s^T[\Sigma^T]\right\},$$
$$\Psi^{T-\varepsilon} := \Psi[\Gamma \times [\varepsilon, T]] = \left\{\psi^h(\,\cdot\,T) \mid h \in \dot{\mathscr{F}}_s^T[\Sigma^T],\ h\big|_{\Gamma\times[0,\varepsilon]} = 0\right\},$$
$$\Psi[\Xi_p^T[\sigma]] = \left\{\psi^h(\,\cdot\,T) \mid h \in \dot{\mathscr{F}}_s^T[\Xi_p^T[\sigma]]\right\}.$$

By (4.20) and (4.21), the fields in $\Psi^{T-\varepsilon}$ generated by retarded controls are localized in the subdomain $\Omega_s^{T-\varepsilon}$. We recall that the subspaces $\mathscr{J}[\dots] \subset \mathscr{J}$ consist of solenoidal fields localized in the corresponding subdomains. By the controllability of (4.26) and (4.27), we have $\overline{\Psi^T} = \mathscr{J}[\Omega_s^T]$, $\overline{\Psi^{T-\varepsilon}} = \mathscr{J}[\Omega_s^{T-\varepsilon}]$, and $\overline{\Psi[\Xi_p^T[\sigma]]} \supset \mathscr{J}[\Omega_s^T[\sigma]]$. From the results of [2] we obtain the equality

$$(\overline{\Psi^T} \ominus \overline{\Psi^{T-\varepsilon}}) \cap \overline{\Psi[\Xi_p^T[\sigma]]} = \mathscr{J}[\omega_s^{T,\varepsilon}] \tag{5.2}$$

(cf. [2, Theorem 3], where a stronger assertion was established).

5.2. Caps in the system α^T. For the system (4.1)–(4.3) we consider the reachable sets

$$\mathscr{U}^T := \mathscr{U}[\Sigma^T] = \left\{u^f(\,\cdot\,T) \mid f \in \dot{\mathscr{F}}^T[\Sigma^T]\right\},$$
$$\mathscr{U}^{T-\varepsilon} := \mathscr{U}[\Gamma \times [\varepsilon, T]] = \left\{u^f(\,\cdot\,T) \mid f \in \dot{\mathscr{F}}^T[\Sigma^T],\ f\big|_{\Gamma\times[0,\varepsilon]} = 0\right\},$$
$$\mathscr{U}[\Xi_p^T[\sigma]] = \left\{u^f(\,\cdot\,T) \mid f \in \dot{\mathscr{F}}^T[\Xi_p^T[\sigma]]\right\}.$$

We note that, by (4.4), the fields in $\mathscr{U}^{T-\varepsilon}$ formed by retarded controls are located in $\overline{\Omega_p^{T-\varepsilon}}$.

Taking into account (5.1) and (5.2), we consider the question about the structure of the subspaces $(\overline{\mathscr{U}}^T \ominus \overline{\mathscr{U}}^{T-\varepsilon}) \cap \overline{\mathscr{U}[\Xi_p^T[\sigma]]}$. The answer to this question is the main result of the paper and reads as follows. We recall that $\mathscr{G}[\dots] \subset \mathscr{G}$ are subspaces of potential fields localized in the corresponding subdomains.

Theorem 1. *Under the assumptions accepted at the beginning of Section* 5*,*

$$(\overline{\mathscr{U}}^T \ominus \overline{\mathscr{U}}^{T-\varepsilon}) \cap \overline{\mathscr{U}[\Xi_p^T[\sigma]]} = \mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]] \oplus \mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]]. \tag{5.3}$$

It is a result we kept in mind, saying about separation of caps in the Lamé type system.

Proof. 1. We show that the right-hand side of (5.3) is a subset of the left-hand side. We have

$$\mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]] = \left\{\nabla q \mid q \in W_2^1(\Omega),\, \operatorname{supp} q \subset \omega_p^{T,\varepsilon}[\sigma]\right\} \overset{(4.15)}{\subset} \overline{\nabla\Phi[\Xi_p^T[\sigma]]} \overset{(4.33)}{\subset} \overline{\mathscr{U}[\Xi_p^T[\sigma]]} \subset \overline{\mathscr{U}^T}.$$

At the same time, $\mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]] \perp \overline{\mathscr{U}}^{T-\varepsilon}$ (in $\mathscr{H}$) since $\omega_p^{T,\varepsilon}[\sigma] \cap \Omega_p^{T-\varepsilon} = \varnothing$ (cf. Figure 1). Consequently,

$$(\overline{\mathscr{U}}^T \ominus \overline{\mathscr{U}}^{T-\varepsilon}) \cap \overline{\mathscr{U}[\Xi_p^T[\sigma]]} \supset \mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]]. \tag{5.4}$$

For the solenoidal subspace on the right-hand side of (5.3) we have

$$\mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]] = \{y \in \mathscr{H} \mid \operatorname{div} y = 0,\ \operatorname{supp} y \subset \omega_s^{T,\varepsilon}[\sigma]\} \overset{(4.26)}{\subset} \overline{\operatorname{rot}\Psi[\Xi_p^T[\sigma]]} \overset{(4.33)}{\subset} \overline{\mathscr{U}[\Xi_p^T[\sigma]]} \subset \overline{\mathscr{U}^T}.$$

At the same time, $\mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]] \perp \overline{\mathscr{U}^{T-\varepsilon}}$. Indeed, from (4.31) (with T replaced by $T-\varepsilon$) it follows that the fields in $\overline{\operatorname{rot}\Psi[\Sigma^{T-\varepsilon}]}$ are located in $\Omega_s^{T-\varepsilon}$. Therefore, in the subdomain $\Omega_p^{T-\varepsilon}\backslash\Omega_s^{T-\varepsilon} \supset \omega_s^{T,\varepsilon}[\sigma]$, the fields in $\overline{\mathscr{U}^{T-\varepsilon}}$ are gradients, which are automatically orthogonal to solenoidal fields located in $\omega_s^{T,\varepsilon}[\sigma]$. Consequently,

$$(\overline{\mathscr{U}^T} \ominus \overline{\mathscr{U}^{T-\varepsilon}}) \cap \overline{\mathscr{U}[\Xi_p^T[\sigma]]} \supset \mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]]. \tag{5.5}$$

Since the caps are at a positive distance from each other, $\mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]] \perp \mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]$. Comparing (5.4) with (5.5), we conclude that

$$(\overline{\mathscr{U}^T} \ominus \overline{\mathscr{U}^{T-\varepsilon}}) \cap \overline{\mathscr{U}[\Xi_p^T[\sigma]]} \supset \mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]] \oplus \mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]]. \tag{5.6}$$

2. We establish the inverse embedding of (5.6).

2a. We show that

$$\mathscr{H}[\Omega_s^T] \subset \overline{\mathscr{U}^T}. \tag{5.7}$$

Let $y \in \mathscr{H}[\Omega_p^T] \ominus \overline{\mathscr{U}^T}$. Then the field y is orthogonal to both terms in (4.32). From $y \perp \mathscr{J}[\Omega_s^T]$ it is easy to obtain the representation $y\big|_{\Omega_s^T} = \nabla\eta$, where $\Delta\eta = 0$ in Ω_s^T and $\eta|_\Gamma = 0$. Since $y \perp \mathscr{G}[\Omega_p^T]$, for any φ with support in $\Omega_s^T \cup \Gamma$, we have

$$0 = \int\limits_{\Omega_s^T} \nabla\eta\cdot\nabla\varphi\,dx = \int\limits_\Gamma \frac{\partial\eta}{\partial\nu}\,\varphi\,d\Gamma.$$

Since $\varphi\big|_\Gamma$ is arbitrary, we have $\dfrac{\partial\eta}{\partial\nu}|_\Gamma = 0$. A harmonic function with the zero Cauchy data $\eta\big|_\Gamma = \dfrac{\partial\eta}{\partial\nu}\Big|_\Gamma = 0$ vanishes identically. Hence $y = 0$ in Ω_s^T. Thus, $y \in \mathscr{H}[\Omega_p^T] \ominus \overline{\mathscr{U}^T}$ implies $\operatorname{supp} y \subset \overline{\Omega_p^T}\backslash\Omega_s^T$, which leads to (5.7).

2b. From (4.32) and $\mathscr{J}[\Omega_s^T] \subset \mathscr{H}[\Omega_s^T] \subset \overline{\mathscr{U}^T}$ we obtain the representation [3)]

$$\overline{\mathscr{U}^T} = \mathscr{H}[\Omega_s^T] \oplus \Big\{y \in \mathscr{H}[\Omega_p^T] \mid \operatorname{supp} y \subset \overline{\Omega_p^T}\backslash\Omega_s^T,\ y\big|_{\Omega_p^T\backslash\Omega_s^T} = \nabla\varphi : \\ \varphi \in W_2^1(\Omega_p^T\backslash\Omega_s^T),\ \varphi|_{\Gamma_p^T} = 0\Big\}. \tag{5.8}$$

Let P^T be an (orthogonal) projection from $\mathscr{H}[\Omega_p^T]$ onto $\overline{\mathscr{U}^T}$. It is easy to derive from (5.8) the representation

$$P^T y = \begin{cases} y & \text{in } \Omega_s^T, \\ \nabla q & \text{in } \Omega_p^T\backslash\Omega_s^T, \end{cases} \qquad \text{where} \qquad \begin{cases} \Delta q = \operatorname{div} y & \text{in } \Omega_p^T\backslash\overline{\Omega_s^T}, \\ q\big|_{\Gamma_p^T} = 0, & \dfrac{\partial q}{\partial\nu}\big|_{\Gamma_s^T} = \nu\cdot y \end{cases} \tag{5.9}$$

3) This representation clarifies the structure of the set $\mathscr{U}^T$ in the subdomain $\Omega_p^T\backslash\Omega_s^T$. A detailed description of the full Lamé system (3.1)–(3.3) is unknown, which causes certain difficulties for studying the corresponding inverse problem [2].

(here, $\nu$is the outward normal to $\partial\{\Omega_p^T\backslash\Omega_s^T\}$).

Similarly, for the projection from $\mathscr{H}[\Omega_p^T]$ onto $\overline{\mathscr{U}^{T-\varepsilon}}$

$$P^{T-\varepsilon}y=\begin{cases} y & \text{in } \Omega_s^{T-\varepsilon}, \\ \nabla r & \text{in } \Omega_p^{T-\varepsilon}\backslash\Omega_s^{T-\varepsilon}, \end{cases} \quad \text{where} \quad \begin{cases} \Delta r=\operatorname{div} y & \text{in } \Omega_p^{T-\varepsilon}\backslash\overline{\Omega_s^{T-\varepsilon}}, \\ r\big|_{\Gamma_p^{T-\varepsilon}}=0, & \dfrac{\partial r}{\partial\nu}\big|_{\Gamma_s^{T-\varepsilon}}=\nu\cdot y \end{cases} \tag{5.10}$$

(ν is the outward normal to $\partial\{\Omega_p^{T-\varepsilon}\backslash\Omega_s^{T-\varepsilon}\}$).

From (5.9) and (5.10) we obtain the relations

$$\left\{\left(P^T-P^{T-\varepsilon}\right)y\right\}\big|_{\Omega_s^{T-\varepsilon}}=0, \quad \left\{\left(P^T-P^{T-\varepsilon}\right)y\right\}\big|_{\Omega_p^{T-\varepsilon}\backslash\overline{\Omega}_s^T}=\nabla(q-r). \tag{5.11}$$

Since $\operatorname{div}\nabla(q-r)=\Delta q-\Delta r=0$ and $\mathrm{r}\,ot\,\nabla(q-r)=0$, the field $(P^T-P^{T-\varepsilon})y$ is harmonic in the subdomain $\Omega_p^{T-\varepsilon}\backslash\overline{\Omega}_s^T$.

2c. Let $y\in(\overline{\mathscr{U}}^T\ominus\overline{\mathscr{U}^{T-\varepsilon}})\cap\overline{\mathscr{U}[\Xi_p^T[\sigma]]}$, so that $y=(P^T-P^{T-\varepsilon})y$. Such a field is harmonic in $\Omega_p^{T-\varepsilon}\backslash\overline{\Omega}_s^T$ and, at the same time, is localized in $\Omega_p^T[\sigma]$. By the known uniqueness theorem for harmonic fields, we conclude that $y=0$ everywhere in $\Omega_p^{T-\varepsilon}\backslash\overline{\Omega}_s^T$. Moreover, from the first relation in (5.11) we find $y\big|_{\Omega_s^{T-\varepsilon}}=0$.

The above considerations lead to the representation $y=y_1\oplus y_2$, where $\operatorname{supp}y_1\subset(\overline{\Omega_p^T}\backslash\Omega_p^{T-\varepsilon})\cap\overline{\Omega_p^T[\sigma]}=\omega_p^{T,\varepsilon}[\sigma]$ and $\operatorname{supp}y_2\subset(\overline{\Omega_s^T}\backslash\Omega_s^{T-\varepsilon})$, whereas the terms are orthogonal because their supports are separated.

Regarding the location of support (in the cap $\omega_p^{T,\varepsilon}[\sigma]$), the field y_1 enters into the first term on the right-hand side of (4.32), i.e., it is a potential field. Thus, $y_1\in\mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]]$.

The restrictions of fields in $\overline{\mathscr{U}^{T-\varepsilon}}$ to the "layer" $\Omega_s^T\backslash\Omega_s^{T-\varepsilon}$ are potential fields with arbitrary potentials on the layer boundary. The field y_2 is localized in the same layer and is orthogonal to $\overline{\mathscr{U}^{T-\varepsilon}}$. Consequently, $y_2\in\mathscr{J}[\Omega_s^T\backslash\Omega_s^{T-\varepsilon}]$. Moreover, $\operatorname{supp}y_2\subset\Omega_p^T[\sigma]$, which implies $y_2\in\mathscr{J}[\{\Omega_s^T\backslash\Omega_s^{T-\varepsilon}\}\cap\Omega_p^T[\sigma]]$.

By (4.33), the fields in $\overline{\mathscr{U}[\Xi_p^T[\sigma]]}$ in the subdomain $\left[\{\Omega_s^T\backslash\Omega_s^{T-\varepsilon}\}\cap\Omega_p^T[\sigma]\right]\backslash\omega_s^{T,\varepsilon}[\sigma]$ are potential. The solenoidal field y_2 also possesses these properties. Hence y_2 is harmonic outside the cap $\omega_s^{T,\varepsilon}[\sigma]$ and vanishes outside $\Omega_p^T[\sigma]$. By harmonicity, it vanishes everywhere outside the cap, i.e., $\operatorname{supp}y_2\subset\omega_s^{T,\varepsilon}[\sigma]$ and, respectively, $y_2\in\mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]]$.

Finally, we conclude that $y=y_1+y_2\in\mathscr{G}[\omega_p^{T,\varepsilon}[\sigma]]\oplus\mathscr{J}[\omega_s^{T,\varepsilon}[\sigma]]$. Thereby the inverse embedding to (5.6) is established. □

5.3. Comments. 1. It is clear that our results are of local character (with respect to x) and can be expanded to the case of unbounded domains. Only the definition of the regularity zone $\Omega^{T^{\text{reg}}}$ should be specified.

2. As was noted in Introduction, we study reachable sets within the framework of the boundary control method. Our final goal is to solve inverse problems [2]. Due to the separation of caps, established by Theorem 1, we may announce the following result: the approach of [1] using virtual sources is extended to a Lamé type system and allows us to restore the velocity c_p (and, possibly, the velocity c_s) at least in a regular zone. We plan to discuss this aspect in our forthcoming publications.

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